Dynamic Analysis of Multi-Body Mechanical Systems with Imperfect Kinematic Joints: A Literature Survey and Review

Onesmus Muvengei, John Kihiu and Bernard Ikua

Abstract—In traditional dynamic modeling of multi-body mechanical systems, the kinematic joints are assumed to be ideal or perfect, i.e., the clearance, friction, wear, and lubrication effects at the joints are neglected in order to simplify the dynamic model. However, in a real mechanical joint, a gap is always present to allow for the relative motion between the connected links as well as to permit the assembly of components. Studies have shown that the clearances and the tribological effects at the joint seriously affect the dynamic response of mechanical systems, and therefore, proper modeling of the imperfect joints in multi-body mechanical systems is required to achieve better understanding of the dynamic performance of the multi-body systems, especially at this era of the increasing demand for high-speed and precise mechanisms and machines.

In this paper a state of the art on dynamic modeling of multi-body mechanical systems with imperfect kinematic joints is critically reviewed. The goal is to review past and recent developments and approaches used in the computer-aided kinematic and dynamic analysis of mechanical systems with real joints. Suitability of various strategies employed in computational dynamics, and also the insufficiently addressed areas which require further attention in this field of study are presented.

Keywords—Cartesian coordinates, Imperfect kinematic joints, Multi-body system

I. Introduction

A MULTI-BODY mechanical system can be defined as an assembly of two or more bodies that are imperfectly connected to each other by use of different types of joints so that the bodies are constrained to move relative to each other. Mechanical systems may range from comparatively simple systems such as slider-crank mechanisms or four-bar linkages to much more complex systems such as those applied in automotive, railway, aerospace, robotics and earth machinery industries, among others.

The bodies which make up a mechanical system can either be rigid or flexible. A rigid body is one which can translate and/or rotate, but can not deform on application of forces. Although a truly rigid body does not exist, many engineering components are assumed to be rigid because their deformations and distortions are negligible in comparison with their relative movements. Hence traditional analysis of multi-body mechanical systems assumes all the links to be rigid. However, many multi-body mechanical systems such as automotive mechanisms, robots, aircraft mechanisms and earth moving machineries which undergo large translation and rotation displacements are either made massive in order to increase rigidity, or are driven slowly so that the dynamic flexibility effects are not significant. As a result, more power is needed to drive them and subsequently, lower work efficiency is achieved. In search for higher efficiency, low cost and greater productivity in machines and mechanisms, links of lighter weights are currently being designed to carry larger loads, and to operate at higher speeds. The use of light weight materials to produce the links of a multi-body system can reduce the driving power and increase the response speed. However, light members are more flexible and hence the standard rigid body mechanism models can no longer be used to accurately predict the behavior of these systems.

The imperfect joining of two bodies that make up a multi-body system is called a kinematic pair or kinematic joint or simply a joint. These joints which include revolute, prismatic, spherical and cylindrical, among others, introduce kinematic constraints to the system by removing some degrees of freedom. In Figure 1, some of the most common types of mechanical joints are shown.

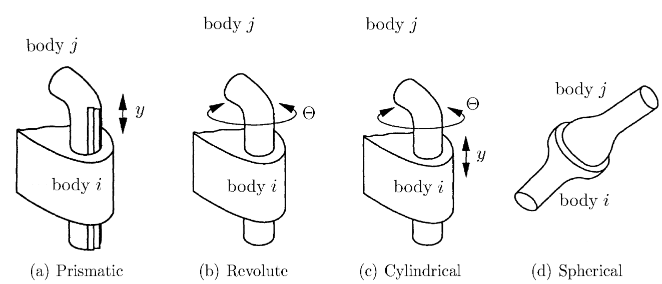

Fig. 1. Types of joints [1]

A prismatic joint allows only relative translation between body *i* and body *j*, while a revolute joint allows only relative rotation between body *i* and body *j*. The cylindrical joint allows body *i* to translate and rotate with respect to body *j* along and about the joint axis, while a spherical joint allows body *i* to rotate with respect to body *j* about all three axes. In planar multi-body systems, revolute and prismatic joints are the ones commonly used to connect the bodies.

Traditionally, the kinematic joints are assumed to be ideal or perfect, i.e., the clearance, friction, wear, and lubrica-

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tion effects at the joints are neglected in order to simplify the dynamic model. However, in a real mechanical joint, a gap is always present to allow the relative motion between the connected links as well as to permit the components' assemblage. These clearances seriously affect the dynamic response of mechanical systems. For instance, impact forces due to clearance induce increased vibration and noise, reduce component life and result in a loss of precision. Therefore, proper modeling of clearance joints in multi-body mechanical systems is required to achieve better understanding of the dynamic performance of the machines and mechanisms.

When studying the multi-body system motion, two different types of analysis are performed, namely, kinematic analysis and dynamic analysis. Kinematic analysis involves the study of the system's motion without considering the forces which produce the motion, i.e., only the position, velocity, acceleration and jerk of system components are determined. Kinematic problems are purely geometrical in nature and can be solved irrespective of the forces and the inertia characteristics such as mass, moment of inertia and the position of the center of gravity of the links. On the other hand, dynamic analysis involves the study of the system motion in relation to the causes which produce the motion, including externally applied forces and moments.

Traditionally, the kinematic solutions are obtained graphically and analytically for simple cases. The forces and moments on the bodies are then obtained based on the kinematic data. These traditional methods are only available for simple cases. However, on advent of high speed digital computers, numerical solutions for complex multi-body systems have been developed. The advantage of computational approach in kinematic and dynamic analysis of a multi-body system is that it can be employed to a system with very many rigid bodies, and makes it easy for the inclusion of some practical features such as friction, beam elasticity, contact compliance and lubrication.

A. Genesis of Multi-body Dynamics

The genesis of multi-body dynamic has its roots at the origin of creation. After creation, man had to use the image of God which he was created with to understand the blank universe he found, and make inventions which will make his life easier. The earliest man found that his own powers were inadequate for the tasks he had to perform on the blank universe to make the world a better place to live. For instance one of the currently acknowledged historical evidence pointing to the ancient technology is the Aztec buildings or the Pharaohs' Pyramids. The practical use of dynamics as a tool must have first taken place with the use of rolling logs for the transportation of heavy stone slabs to the construction sites and the hoisting of these with pulleys. These early inventions seen as utilizing dynamic principles, must have occurred along with the use of wheels, which coupled with a cart, a horse and a driver, must represent the first multi-body dynamic system [2]

It took thousands of years for man to find a scientific explanation for mechanical phenomena, with the first known attempts on the Fourth century BC. The mechanical devices (the lever, pulley, etc.) of the time were extremely simple and were mainly studied from the standpoint of static equilibrium of forces, and the knowledge of mechanics was limited. Some of the most important work in the field of static was done by Archimedes (287-212 BC), who carried a research on the laws of the lever, center of gravity, among other phenomena. It was his work on the lever which caused Archimedes to remark, *"Give me a place to stand on, and I will move the Earth"*. Archimedes designed block-and-tackle pulley systems, allowing sailors to use the principle of leverage to lift objects that would otherwise have been too heavy to move. Archimedes is also credited with improving the power and accuracy of the catapult, and with inventing the odometer during the First Punic War. The odometer was described as a cart with a gear mechanism that dropped a ball into a container after each traveled mile in order to measure distance. All these discoveries were aimed to the defense of Syracuse, his motherland, against the Roman siege in the First and Second Punic Wars.

After Archimedes, there was little advance in mechanics until the Fifteenth century AC, when it began to develop intensively with Leonardo de Vinci (1452-1519) an Italian, making several discoveries in the field of mechanics such as the moment and the concept of work of a force.

Before publication of Newton's Principles (1687), dynamics was an empirical science, that is, it considered description of observed behavior without an explanation for the causes that produced motion. Galileo (1564-1642) performed experiments using pendulums and realized that, a pendulum swing is constant regardless of amplitude (at least to small angle approximations). In 1592 he took the trouble of dropping balls from the top of the leaning Tower of Pisa to ascertain that the rate of fall is constant for all balls of different mass, thereby discovering the acceleration of free fall. This finding contradicted the widely held Aristotelian belief that speed of fall is proportional to weight, and he was dismissed from his position in the University of Pisa.

The classical mechanics theory as it is known today began when Newton (1642-1727) introduced the law of gravitation and explained the concepts of force, momentum, and acceleration [3]. Shortly after Newton's laws were formulated, important techniques for their application were developed by Euler, D'Alembert, Lagrange, Hamilton, Coriolis, Einstein among others.

Since much of Newton's work was geometric, Euler (1707-1783) extended the original ideas of Newton to include rigid body motions and restated it using calculus. The concept of dynamic equilibrium was firstly postulated by D'Alembert (1717- 1783), who introduced the concept of inertia force. This concept, coupled with the result of Newtons work, yields an equation stating that the sum of the force on a body in motion is zero if the inertia force is included. This seemingly simple fact has very important implications in applied mechanics. The inclusion of calculus in dynamic equilibrium was systemized by Lagrange (1736-1813), who analytically derived the generalized equations of motion by using energy concept. In addition, Lagrange devised a formulation for dynamics of constrained multi-body systems. This contribution opened the way for formulation and solution of complex and practical mechanisms, making Lagrange the father of multi-body dynamics. The Lagrange formulation method is so robust and generic that it has not only withstood the test of time, but also has required no significant modification or advancement since its inception in 1788 [2]. Additions to multi-body dynamics theory since then have been in the development of more efficient numerical solutions of what has rightly become known as Lagrangian dynamics, or in the inclusion of some practical features such as friction, beam elasticity, contact compliance and lubrication. The inclusion of these additional features has paved the way for the analysis of complex and practical modern machines and mechanisms.

With the advent of ever increasing computing power, analysis of practical multi-body dynamic problems has become possible. Using the principles of Galileo and Lagrange, but with numerical integration methods for stiff systems, it is possible to analyze not only mechanisms as simple as the swinging chandelier, which roused the curiosity of Galileo, but also complex multi-body systems.

II. Multi-body Systems with Imperfect Kinematic Joints

A. Prior Research

Over the last three decades, the dynamic modeling of multi-body systems has been recognized as a key aid in the analysis, design, optimization, control, and simulation of mechanisms and manipulators. However, clearance, friction, impact and other phenomena associated with imperfect joints are routinely ignored. Therefore, the dynamics of mechanical systems is often conducted under the assumption that, the joints are ideal or perfect. The increasing requirement for high-speed and precise machines, mechanisms and manipulators demands that the joints be treated in a realistic way, i.e., the joint clearances, friction and the lubrication effect need to be considered when determining the dynamic model of the mechanical system. This is because in a real mechanical joint, a clearance which allows for the relative motion between the connected bodies as well as permitting the components assemblage, is always present. The clearance no matter how small it is, can lead to vibration and fatigue phenomena, premature failure and lack of precision or even random overall behavior. If the joint is not effectively lubricated, the impacts occurring in the system and the corresponding impulses are transmitted throughout the mechanical system [4].

There is a significant amount of literature available which discuses theoretical and experimental analysis of imperfect kinematic joints in a variety of planar and spatial mechanical systems with rigid or flexible links. Many of these works focus on the planar systems in which only one kinematic joint is modeled as an imperfect joint. Dubowsky *et al.*(1971) [5], [6] formulated an impact model by arranging springs and dashpots as KelvinVoigt model to predict the dynamic response of an elastic mechanical joint with clearance, which was later extended in [7] to include flexible mechanisms with clearance connections.

Earles and Wu (1973) [8] employed a modified Lagrange's equation approach in which constraints were incorporated using Lagrange multipliers in order to predict the behavior of rigid mechanism with clearance in a journal bearing. The authors modeled the clearance in the journal bearing as a massless imaginary link whose length was equal to the clearance size. In their further work (1977) [9], the authors used the model to predict contact loss between the joint components for a planar mechanism. The concept of a massless link was also used by Furuhashi *et al.* (1978) [10]–[13] to study the dynamics of a four-bar linkage with clearance at revolute joints using continuous contact models. Once again, the joint components were assumed to be in contact at all times.

More complex models have also been developed to study the effect of clearance on multi-body system dynamics. Considering the joint to consist of two components, Farahanchi and Shaw (1994) [14] modeled joint clearance by considering three configurations of the joint components: Free-flight motion, when the components are not in contact; the impact condition, when the components establish contact; and the sliding condition, when the components are in contact and in relative motion. The authors used a slider-crank mechanism to demonstrate their procedure and studied the effect of clearance size, friction, crank speed and impact parameters. Rhee *et al.* (1996) [15] also used the three modes of motion to model the joint clearance by using an approach similar to that of Farahanchi [14] to determine the reaction force during the sliding motion and to study the response of a four-bar mechanism with clearance in a revolute joint.

Ravn (1998) [16] also implemented the three mode approach to model the joint clearance. However, in his approach, the reaction force during the impact and sliding model was computed using a contact force model. This analysis has been termed as the continuous method since integration of the equations of motion is not halted as in the case of the discontinuous method. A number of recent researchers (2000-2010) [17]–[31] have since used this technique to model and study the effect of clearance in the joints of multi-body systems.

Many papers available have focused on mechanical systems with rigid links, but this assumption seems to be too restrictive for many mechanical systems. However, link flexibility effects have been considered in a few investigations. Dubowsky *et al.* (1987) [7] considered the effect of link flexibility for both planar and spatial mechanical systems but ignored the effects of friction and lubrication at the joint. Liu *et al.* (1990) [32] added the lubrication effects through the squeeze film formula and Reynolds number, but their formulation was limited to planar problems. Ravn *et al.* (1999) [33] studied the effect of joint clearance on system response including the effect of lubrication and link flexibility. However, their model was limited to planar problems and they modeled only one kinematic joint in a system as an imperfect joint. Olivier *et al.* (2002) [34] studied the effect of revolute and spherical joint clearance, lubrication and link flexibility on the overall dynamic behavior of systems. However, the authors modeled only one kinematic joint in a system as an imperfect joint. Dubowsky and Moening (1978) [35] obtained a reduction in the impact force level by introducing bodies' flexibility. They also observed a significant reduction of the acoustical noise produced by the impact when the system incorporates flexible bodies. Kakizaki *et al.* (1993) [36] presented a model for spatial dynamics of robotic manipulators with flexible links and joint clearances, where the effect of the joint clearance was used to control the robotic system. Chunmei *et al.* (2002) [18] observed that, the stability of a multi-body system with joint clearances is far much lower than a system with no clearances. In addition, the authors found that the consideration of link flexibility in a multi-body system can reduce the peaks of impact forces to some degree. Zongyu Chang (2007) [19] studied the nonlinear dynamic behavior of a four-bar linkage with an imperfect revolute joint between the coupler and the follower links. The authors showed that, a clearance in multi-body system causes periodic and chaotic responses depending on the coefficient of friction and the clearance size. Jia *et al.* (2002) [20] presented both theoretical and experimental studies about dynamic behavior of a slider-crank mechanism with clearance at the connecting rod and slider joint. The authors also investigated the effects of different clearance size and driving speed on dynamic behavior of the mechanism. Schwab *et al.* (2002) [21] investigated the dynamic response of mechanisms and machines having a revolute joint with clearance but without friction. Assuming that a mechanism consists of rigid and elastic links, a comparison between several continuous contact force models and an impact model were also presented. In addition, the authors also presented a procedure to estimate the maximum contact force during impact. Flores *et al.* (2004) [22] presented dynamic analysis of multi-body systems with revolute joint clearances, including dry contact and lubricant effects. Flores and Ambrosio (2004) [23] presented a general methodology for dynamic characterization of mechanical systems, in which revolute joints have clearance. The authors used a slider-crank mechanism with an imperfect joint to demonstrate the contact detection strategy and the contact force models they proposed in their procedure.

More recently, Erkaya and Uzmay (2010) [24] investigated theoretically and experimentally the effects of joint clearances on vibration and noise characteristics of a slider-crank mechanism with two imperfect revolute joints. The authors modeled the joints clearance as massless virtual links and assumed a continuous contact mode between journal and bearing in the joints connection. They observed that; joint clearance leads to a degradation in vibration characteristics of the mechanism relative to that of mechanism without clearance, vibration characteristics is mainly periodic since continuous contact between journal and bearing at a joint is normally assumed, and noise level of the mechanism with clearance is higher than that of the mechanism without clearance. Although this research work considered two imperfect revolute joints unlike many papers in literature, the authors only considered rigid link dynamics and did not investigate the interaction effects of these real joints on the overall response of a multi-body system. Shiau *et al.* (2008) [25] studied nonlinear dynamic analysis of a 3-PRS series parallel mechanism considering the flexibility of links, clearance and friction at the revolute joints. The authors applied the Newtonian approach to derive the equations of motion of the 3-PRS mechanism, and combined the Runge-Kutta method and contact verification criterion to solve the nonlinear differential equations. The obtained results showed that the joint clearance significantly affects the mode shapes by which the rotational motions are dominated. Some of the natural frequencies decrease as the joint clearance increases. The dynamic response also becomes larger as the joint clearances increase, and the contact force increases as the joint clearance and the friction coefficient increase. The effects of joint clearances on mechanism path generation and transmission quality have been investigated by Erkaya and Uzmay (2009) [26]–[28]. The authors considered four-bar and slider-crank mechanisms having joints with clearance as model mechanisms, and proposed an optimization procedure for decreasing the deviations of path generation and transmission angle. Khemili and Romdhane (2008) [29] investigated theoretically and experimentally the dynamic behavior of a planar slider-crank mechanism with an imperfect revolute joint between the connecting rod and the slider. By using a contact model based on the so-called impact-function, the authors developed a model for the simulation tests under the software ADAMS, and constructed an experimental rig to achieve some experimental validations. The authors observed the occurrence of three types of motion at the clearance, that is, a free-flight, a continuous contact motion and an impact. In addition, the authors showed that in case of a flexible link, the impact forces, the slider acceleration values and the driving torque at the crank are lower. Hence they concluded that the flexibility of a link plays a role of suspension for the multi-body mechanical system. Erkaya and Uzmay (2009) [30] investigated dynamic response of a four-bar mechanism having two revolute joints with clearance. The authors proposed a neural-genetic approach to model several characteristics of joint clearance and determine the appropriate values of design variables for reducing the additional vibration effects due to the clearances. Their results showed that the optimum adjusting of suitable design variables gives a certain decrease in shaking forces and their moments on the mechanism frame. Although this research work considered two imperfect revolute joints unlike many papers in literature, the authors only considered rigid link dynamics and did not investigate the interaction effects of these real joints on the overall response of a multi-body system. Bing and Ye (2008) [31] presented dynamic analysis of the reheat-stop-valve mechanism with revolute clearance joint. The authors analyzed the effect of joint clearance variation induced by the manufacturing tolerance of components combined with the thermal influence of the high temperature steam in working condition. Flores (2009) [37] developed a methodology for studying and quantifying the wear phenomenon in revolute clearance joints. The author used a simple model for a revolute joint in the framework of multi-body systems formulation, and based the contact forces on a continuous contact force model. Also, friction effects due to the contact in the joints were represented. Then these contact-impact forces were used to compute the pressure field at the contact zone, which ultimately has been employed to quantify the wear developed and caused by the relative sliding motion. In addition, the author used simple planar multi-body mechanical system in form of a four bar linkage with one imperfect joint to perform numerical simulations. His results showed that the wear phenomenon is not uniformly distributed around the joint surface, owing to the fact that the contact between the joint elements is wider and more frequent in some specific regions. Dupac and Beale (2010) [38] investigated the dynamics and stability of a planar slider crank mechanism with a flexible rod with cracks and a slider clearance. The authors developed the equations of motion in which the influence of the cracks size, slider clearance and impact effects were jointly considered. They observed that the dynamic behavior of the mechanism is significantly changed by the effects of clearance combined with the imperfect links, and hence these should never be ignored when analyzing the dynamic performance and vibration characteristics of mechanical systems. Flores *et al*. (2009) [4] presented a general methodology for modeling lubricated revolute joints in constrained rigid multi-body systems. The authors obtained the hydrodynamic forces by integrating the pressure distribution evaluated with the aid of Reynolds equation, written for the dynamic regime. Numerical examples were presented in order to demonstrate the use of the methodologies and procedures described in the work. Mukras *et al.* (2010) [39] presented a procedure to analyze planar multi-body systems experiencing wear at a single revolute joint. The analysis involved modeling multi-body systems with revolute joints that consist of clearance and then incorporating wear into the system dynamic analysis by allowing the size and shape of the clearance to evolve as dictated by wear. The authors used an iterative wear prediction procedure based on Archards wear model to compute the wear as a function of the evolving dynamics and tribological data. The procedure was then validated by comparing the wear prediction with wear on an imperfect joint of an experimental slider-crank mechanism.

B. Impact

Impact is a complex phenomenon that occurs when two or more bodies collide with each other [40]. This phenomenon is important in many different areas such as machine design, robotics, multi-body analysis, just to mention but a few.

Impact, which occurs due to the collision at the joint (because of the clearance), is the most widely studied effect of imperfect joints on the dynamic behavior of mechanical multi-body system. As a result of impact, it has been shown that the mechanical system state responses change rapidly, and the velocities and accelerations appear to have some discontinuities. The impact is characterized by large forces that are applied and removed in a short period of time. The knowledge of the peak forces generated during the impact process is very important in the dynamic analysis and hence design of a multi-body system. Other effects directly related to the impact phenomena are the vibration and load amplification/propagation to the system components, fatigue, wear at the contact zone and energy dissipation. Therefore, the choice of most adequate and suitable contact-impact force model plays a key role in the correct design and dynamic analysis of the mechanical systems with imperfect kinematic joints.

The impact process is considered to occur in two phases; the compression (or loading) phase and the restitution (or unloading) phase. During the compression phase, the two bodies deform in the direction perpendicular to the surfaces and the relative velocity of the contacting surfaces in that direction is gradually reduced to zero. The end of the compression phase is referred to as the instant of maximum compression. The restitution phase begins at this instant and ends when the two bodies separate from each other. The restitution coefficient reflects the type of collision: For a fully elastic impact, the restitution coefficient is equal to one, while for a fully plastic impact, the restitution coefficient is equal to zero [41].

In order to fully model the contact-impact forces resulting from the collision in multi-body systems with clearance at the joints, information on the impact velocity, material properties of the colliding bodies and geometry characteristics of the contacting surfaces must be included in the model [41]. Three methods which include stress wave propagation, discontinuous and continuous (contact mechanics) methods, have been used to solve the impact problem due to clearance joints in multi-body mechanical systems [42].

1) Stress Wave Propagation Method: When impact occurs, a stress wave is set up in the bodies involved in the collision. In this method, the impact process is studied by analyzing stress waves created and their interactions with the impacting bodies. Although this method is accurate, its not widely used due to its complexity [43].

2) Discontinuous (or Discrete) Formulation Method: Due to the fact that impacts occur very rapidly, several authors have ignored the events that occur during the impact process. Only the states of the system before and after impact are considered to be of interest when determining the coefficient of restitution [44]–[46]. In this method, the coefficient of restitutions are solved by using the linear impulse momentum, the angular impulse momentum, and the relations between the variable before and after impact [41], [47]. This method is normally used when the impact involves very hard and compact bodies. Discontinuous models usually employed are, Poisson's model [48], Newton's model [48] and Stronge's model [49], [50]. These three models are equivalent if friction is not considered, Newton's model neglects slip direction, but Poisson's and Stronge's models dissipate more energy than the Newton's model [51].

Although the discontinuous contact-impact force models are simple, the assumption that the impact duration is very small is not true in many cases, and the use of such models tend to provide poor results when used directly to calculate impact forces [52]. In addition, these discrete models do not work well with Coulomb's empirical laws of friction [53] and are not straightforward when applied in impacts of flexible bodies [54]

3) Continuous Formulation Method: The continuous models are also called compliant contact models and overcome the problems associated with the discrete models

In continuous methods, its assumed that the impact duration is much longer than the fundamental frequencies of the impacting bodies. Hence the forces and deformations arising from the impact process are considered to vary in a continuous manner. In this approach, when contact between the bodies is detected, a normal force perpendicular to the plane of collision is applied. This force is typically applied as a spring-damper element which can be linear or non-linear. The spring represents the elasticity of the contacting bodies while the damper describes the loss of kinetic energy during the impact. The spring stiffness in the element can be calculated using a simple mechanical formula or obtained by means of the Finite Element Method. Zhu *et al.* [55] proposed a theoretical formula for calculating damping in the impact of two bodies in a multi-body system. When the contact bodies are separating from each other, the energy loss is included in the contact model by multiplying the rebound force with a coefficient of restitution.

Kelvin-Voigt, Maxwell and Standard Linear Solid models [56] are the three linear spring-damper configurations which have been used to model the contact-impact forces continuously.

(a) *Kelvin-Voigt Model*- This is the simplest contact force relationship and has a two-parameter parallel linear spring-damper configuration. For this model, the stress-strain relationship is given as;

$$\sigma = E_1\epsilon + c\frac{d\epsilon}{dt} \tag{1}$$

(b) *Maxwell Model*- This is another two-parameter model, with the spring and damper in a series configuration. For this model, the stress-strain relationship is given as;

$$\sigma + \frac{c}{E_1}\frac{d\sigma}{dt} = c\frac{d\epsilon}{dt} \tag{2}$$

(c) *Standard Linear Solid Model*- This is a three-parameter model which uses two springs and a damper. For this model, the stress-strain relationship is given as;

$$\sigma + \frac{c}{E_2}\frac{d\sigma}{dt} = E_1\epsilon + \frac{c}{E_2}(E_1 + E_2)\frac{d\epsilon}{dt} \tag{3}$$

The normal contact force (F_N) in the three linear spring-damper models is calculated for a given penetration depth (δ) as;

$$F_N = K\delta \qquad \text{if } v_N > 0 \text{ (compression phase)}$$
$$F_N = K\delta c_e \qquad \text{if } v_N < 0 \text{ (restitution phase)}$$

where K is the spring stiffness, δ is the relative penetration depth, c_e is the coefficient of restitution and v_N is the relative normal velocity of the colliding bodies. The limitation of these linear models is the calculation of the spring constant, which depends on the geometry and the physical properties of the contacting bodies. In addition, the assumption of a linear relation between the penetration depth and the contact forces is a rough estimation. This is because the contact force depends on the shape, conditions and material properties of the contacting surfaces, all of which suggest a more complex and nonlinear relation. Hunt and Crossley [57] showed that the linear spring-damper models do not represent the physical nature of energy transferred during the impact process.

The Hertz law of contact is the best known nonlinear contact force law commonly used for sphere-to-sphere contact. The Hertzian law is based on the theory of elasticity [58] and is restricted to frictionless surfaces and perfectly elastic solids. The Hertz model relates the contact force as a nonlinear power function of the penetration depth. The force deformation relationship is given as;

$$F_N = K\delta^n \tag{4}$$

where F_N is the normal contact force, δ is the deformation of the contacting bodies, exponent $n = 1.5$ for metallic surfaces and the generalized stiffness K which depends on the material properties and the shape of the contacting surfaces is given as;

$$K = \frac{4}{3(\sigma_1 + \sigma_2)}\Big[\frac{R_1R_2}{R_1 + R_2}\Big]^{\frac{1}{2}} \tag{5}$$

where;

R_1 and R_2 are the radii of the spheres (the radius is negative for concave surfaces and positive for convex surfaces)

σ_1 and σ_2 are the material parameters given by;

$$\sigma_i = \frac{1 - \nu_i^2}{E_i} \qquad \text{for } i = 1, 2$$

where E_i and ν_i are the Young's Modulus and Poisson's ratio associated with each sphere.

Unfortunately, the Hertz Law as given in equation 4 does not account for energy dissipation during the impact process and hence cannot be used in both phases of contact (compression and restitution). Although, some studies have been performed to include the energy dissipation in this contact force law. Lankarani and Nikravesh [59] extended the Hertz contact force model to include a hysteresis damping function to represent the energy dissipated during the impact. The authors separated the normal contact force given in equation 4 into elastic and dissipative components as;

$$F_N = K\delta^n + D\dot{\delta} \tag{6}$$

where $\dot{\delta}$ is the relative impact velocity and D is the hysteresis coefficient given as;

$$D = \Big[\frac{3K(1 - c_e^2)}{4\dot{\delta}^{(-)}}\Big]\delta^n \tag{7}$$

where $\dot{\delta}^{(-)}$ is the initial impact velocity. Therefore the final normal contact force can be expressed as;

$$F_N = K\delta^n\Big[1 + \frac{3(1 - c_e^2)\dot{\delta}}{4\dot{\delta}^{(-)}}\Big] \tag{8}$$

Equation 8 is only valid for impact velocities lower than the propagation velocity of elastic waves across the bodies, i.e., $\dot{\delta} \leq 10^{-5}\sqrt{\frac{E}{\rho}}$ where E is the Young's modulus and ρ is the material mass density [60]. Shivawamy [61] studied theoretically and experimentally the impact between bodies and demonstrated that at low velocities, the hysteresis damping is the key factor for energy dissipation, but at high velocities exceeding the propagation velocity of elastic waves, energy is dissipated in a more complicated form.

4) Contact Force Models for Cylindrical Surfaces: The contact models given by equations 4 and 8 are applicable for colliding bodies with spherical contact areas. For the case of cylindrical contact forces, some authors suggest the use of equation 8 but with an exponent n in the range of 1 to 1.5 [16], [57]. Dietl [62] used the classical solution of contact, presented by Hertz but with exponent n equal to 1.08 to model the contact between the journal-bearing elements.

Various models have been put forward for the cylindrical contact surfaces, such as the Dubowsky and Fruedenstein model reviewed by Flores et.al in [63] as well as the ESDU-78035 model, both of which are given as equations 9 and 10 respectively;

$$\delta \ = \ F_N\Big(\frac{\sigma_1+\sigma_2}{L}\Big)\Big[\ln\Big(\frac{L^3(R_1-R_2)}{F_N R_1 R_2(\sigma_1-\sigma_2)}\Big)+1\Big] \quad (9)$$

and

$$\delta \ = \ F_N\Big(\frac{\sigma_1+\sigma_2}{L}\Big)\Big[\ln\Big(\frac{4L(R_1-R_2)}{F_N(\sigma_1+\sigma_2)}\Big)+1\Big] \quad (10)$$

where L is the length of the cylinder. Equations 9 and 10 are nonlinear function for F_N and require an iterative scheme, such as Newton-Raphson method to solve for the normal contact force F_N for a known penetration depth δ.

C. Link Flexibility

Another phenomenon of interest in the study of imperfections in multi-body system is the flexibility of links. A body is assumed to be rigid if any pair of its material points does not present relative displacements. In practice, bodies suffer some degree of deformation. The deformation tends to be so small that it does not affect the system's behavior, and therefore it can be neglected without committing an appreciable error. There are some important cases in which deformation plays an important role in the dynamic analysis, for instance in lightweight links, or in high-speed machinery. The complexity and size of the equations of motion considering flexibility grow considerably, since all the variables defining the deformation must also be considered in the equations [1].

Studies in link flexibility can be broadly classified as those using modal and nodal coordinates. The advantage of the modal approach is the small number of degrees of freedom, thus reducing the computational burden considerably compared to the nodal approach. The modal method, however requires the pre-calculation of the mode shapes to be used in before the simulations and is not commonly used compared to the nodal approach due to the following reasons:

(a) The choice of the vibration modes is not easy, and the quality of simulation results obtained (besides numerical aspects) depends exclusively on the quality of the mode shapes used in the simulations, i.e., on how accurate the mode shapes can represent the real deformations [64].

(b) The vibration frequencies and modes change as the flexible body undergoes large rotation motion [65].

For further and more information on comparison of nodal and modal coordinate systems formulations in flexible multi-body system dynamics, any interested leader is advised to refer to [64]

Craig [66] proposed a method that has been used to represent or model the responses of a flexible body to an applied load by superimposing several of its modes. Gonclaves and Ambrosio [67] used this method to model flexible body systems in order to study road vehicle comfort. Also, Rokach [68] used this method for one point bend test modeling.

The nodal coordinates approach can be used in two ways, namely, use of lumped rigid segments and springs, and the use of Finite Elements. The lumped rigid segments and spring method divides a deformable body into a number of rigid segments and springs according to the geometric parameters of the body. The formulation for rigid multi-body systems which is developed using Kane's equations, is the applied [69]. Finite Element Method (FEM) which employs nodal coordinates has been proposed as an efficient way of modeling the dynamics of flexible links by treating the flexible links as Euler-Bernoulli beams, i.e., deflection of the beams is due to bending only and hence shear deformations are neglected [70].

FEM involves decomposing a beam into several simple pieces called elements. The elements are assumed to be interconnected at certain points known as nodes. For each element, an equation describing the dynamic behavior of the element is obtained through an approximation technique. The elemental equations are then assembled to form an overall system equation. By reducing the element size, i.e., increasing the number of elements, the overall solution of the system equation can be made to converge to the exact solution.

The Finite element approach for the kinematic and dynamic analysis of mechanisms and machines was initiated by Besseling [71]. In his study of discretization methods for deriving finite dimensional element models for the mechanics of continua, he pointed at the complete analogy between the discrete case and the continuous case.

Dumbrowski [72] presented a formulation employing nodal coordinates to model flexible bodies by using Euler-Bernoulli beam formulation to model bodies undergoing large deformation. Seo et al. [73] also presented a nodal coordinate formulation employing two-dimensional beam elements to represent large deformation flexible bodies.

1) Dynamic Analysis of Flexible Multi-body Systems: Various dynamic approaches have been employed to formulate the equations of motion of flexible multi-body systems, of which the most widely used ones being the Floating Frame of Reference Formulation (FFRF) and Absolute Nodal Coordinates Formulation (ANCF).

Floating Frame of Reference Formulation (FFRF): In the FFRF, the equations of motion are expressed in terms of a coupled set of absolute cartesian and local elastic coordinates. The absolute cartesian coordinates define the location and the orientation of a selected body coordinate system, while the local elastic coordinates define the deformation of the body with respect to its reference. The elastic coordinates can be introduced using component mode methods, the finite element method or experimental identification techniques.

The use of elastic coordinates defined in flexible body coordinate system in small deformation problems leads to a simple and constant stiffness matrix, and allows the use of modal reduction techniques that significantly reduce the dimensionality problem by eliminating insignificant high frequency modes of vibration. This makes the numerical integration of the generated equations of motion more efficient. In addition, the choice of coordinates in the FFRF leads to a consistent and easy to implement formulation when non-isoparametric elements such as beams, plates and shells are considered. The use of this formulation leads to an exact modeling of the rigid body dynamics, and also leads to zero strain under an arbitrary rigid body motion. However, this formulation leads to a highly nonlinear mass matrix as a result of the inertia coupling between the reference motion and the elastic deformation.

The FFRF is the most widely used method in the dynamic simulation of flexible multi-body systems, and has proven to be efficient in handling problems which contain rigid and flexible bodies where small deformation assumptions remain valid. Crucial to the successful and efficient computer implementation of the FFRF is the identification of the body inertia shape integrals. These inertia shape integrals can be evaluated in a pre-processor computer program using lumped or consistent mass approach, and can be expressed in a modal form [74].

Absolute Nodal Coordinates Formulation (ANCF): Despite the success and popularity of the FFRF, its use has been generally limited to the analysis of small deformations. This is due to two reasons [75]:

(a) In order to justify the use of linear modes for coordinate reduction in FFRF, the deformations are assumed to be small.
(b) The use of infinitesimal incremental techniques which utilize infinitesimal rotation as nodal coordinates requires linearization of kinematic equations and does not lead to exact modeling of rigid body dynamics. Hence these techniques are widely employed in structural analysis but can not be employed to analyze small deformations

In order to obtain accurate representation of large deformations in flexible multi-body dynamics, the ANCF has been proposed. In this formulation, global displacement and slope coordinates are used to define the configuration of finite elements. By using global slopes instead of finite and infinitesimal rotations, exact modeling of the rigid body dynamics can be obtained. Because in ANCF the nodal coordinates are defined in the global system, then the use of modal reduction techniques becomes impractical and as a consequence, the large dimensionality becomes a problem in this formulation.

Choice of the Coordinates: Any dynamic solution procedure used in flexible multi-body simulations as well as the degree and type of nonlinearity of the formulation depend on the choice of coordinates. Different sets of coordinates may lead to different structures for the dynamic equations, however the proper selection of the coordinates must take into consideration the type of the problem to be investigated. The use of one set of coordinates as compared to other sets, may prove more efficient in a particular application.

Generally, the choice of a coordinate system in formulation of equations of motion of flexible multi-body system leads to fundamental problems related to the selection of the deformable body coordinate system, the boundary conditions and the appropriate set of mode shapes. Despite these fundamental problems, the FFRF remains the most widely used and accepted procedure in flexible multi-body simulations due to the following reasons [75]:

(a) Many of multi-body applications consists of many rigid bodies and few flexible bodies. There is therefore a tendency not to compromise the accuracy of the rigid body analysis, and to use a formulation which is more efficient in the simulation of the rigid bodies. This can easily be accomplished by using the FFRF.
(b) The concept of angular velocity is fundamental in the analysis of multi-body systems particularly when prescribed motion trajectories are considered. In this case, its much easier to define the angular velocities when FFRF is used.
(c) The FFRF is an extension of the Newton-Euler formulation which has been extensively used in the analysis of multi-body systems consisting of rigid bodies.
(d) Its much easier to develop recursive equations when the FFRF is used. Therefore, the computational implementation of FFRF algorithm is very easy and straightforward.

However, these reasons behind the popularity and acceptance of FFRF by the multi-body community do not make FFRF a superior formulation as compared to the ANCF, but the appropriate choice of either depends entirely on the type of the problem being addressed. Gerstmayr *et al.* [76] carried out a detailed comparison of the ANCF and the FFRF for standard static and dynamic problems, both in the small and large deformation regimes. The authors observed that both formulations have comparable performance, and that the choice of the optimal formulation depends on the problem configuration. Thus, the authors refuted the recent claims in literature that the ANCF would have deficiencies compared with the FFRF. Shabana [75] used a flexible multi-body four-bar mechanism and a flexible pendulum to investigate the performance of FFRF, the incremental method and ANCF. He observed that the results obtained using the three methods agree well in the case of small deformations and low angular velocities. In the case of large deformations and low angular speeds, the author observed that the solutions obtained using the ANCF and incremental method agree well, but at large deformations and higher angular speeds, there was significant differences between the solutions obtained using the two formulations due to the effect of linearization used in the incremental method.

D. Friction

Another phenomenon of interest in the study of imperfections in multi-body system is the dry friction in the kinematic joints. When contacting bodies slide or tend to slide relative to each other, forces are generated which are tangential to the surfaces of contact. These forces are usually referred to as friction forces. Three basic facts about friction have been experimentally established as follows; the friction force is proportional to the normal load on the contact, the friction force acts in a direction opposite to that of the relative motion between the two contacting bodies, and the friction force is independent of nominal area of contact [77]. Luminaries of science such as Coulomb developed friction laws, but there is still no simple model which can be universally applied by engineers to calculate the friction force for a given pair of bodies in contact [78]. In the past three decades, there has been much interest on the study of friction, and there are many research papers which focus on the subject [48], [79], [80]

Coulomb friction law of sliding friction represents the most fundamental and simplest model of friction between dry contacting surfaces. When sliding takes place, the Coulomb law states that the frictional force is directly proportional to the magnitude of normal force at the contact point, where the constant of proportionality is termed as the coefficient of friction. The original Coulomb's friction law is independent of relative tangential velocity, which in practice is not true since friction forces have been shown to depend on many factors such as material properties, temperature, surface cleanliness and velocity of sliding. Moreover, the original Coulomb's friction law does not cater for the stiction phenomena which occurs when the relative tangential velocity of two impacting bodies approaches zero. As pointed out by Ahmed et al. [81], a suitable friction model must be able to detect sliding and sticking to avoid energy gains during impact. Indeed, the classical Coulomb dry friction model does not portray the important physical phenomena which occurs in the contact between mating surfaces. Moreover, the discontinuity of force at zero velocity has many drawbacks during numerical simulations [82]. These drawbacks of classical Coulomb friction model have led several researchers on multi-body dynamics [83]–[87] to modify the Coulomb law in order to avoid the discontinuity of force at zero relative velocity and to obtain a continuous friction force-velocity relationship. Flores [78] used a modified Coulomb's friction law proposed by Ambrosio [87] which gives the tangential friction force (F_T) as;

$$F_T = -\mu c_d F_N \frac{\mathbf{v}_T}{v_T} \tag{11}$$

where μ is the coefficient of friction, F_N is the normal contact force, $\mathbf{v}_T$ is the relative tangential velocity, v_T is the tangential velocity and c_d is a dynamic correction coefficient expressed as,

$$\begin{aligned} c_d &= 0 && \text{if} \quad v_T \leq v_0 \\ &= \frac{v_T - v_0}{v_l - v_0} && \text{if} \quad v_0 \leq v_T \leq v_l \\ &= 1 && \text{if} \quad v_T \geq v_l \end{aligned}$$

in which, v_0 and v_l are the given tolerances for the velocity. The correction factor prevents the friction force from changing direction for almost zero values of the tangential velocity. The advantage of this modified Coulomb's law is that, it allows numerical stabilization of the integration algorithm. However, it does not account for stiction phenomena of the contacting surfaces. This led Flores [78] to recommend that friction laws be developed which account for stick-slip condition in imperfect joints of multi-body mechanical systems.

III. Modeling Multi-body Systems

Computational dynamic analysis of any mechanical system whether with ideal or real joints involves solving numerically the equations of motion of the system for the responses. The process of formulating the equations which govern the behavior of the system is called modeling the system, while the process of numerically solving the generated equations of motion in order to analyze the system's response is termed as simulation. In this section the procedures and approaches which have been employed in generating and computationally solving the equations of motion of a multi-body system are reviewed.

A. Coordinate Systems

The first problem encountered at the time of modeling the motion of a multi-body system is that of finding an appropriate system of coordinates which will allow one to clearly define at all times the position, velocity and acceleration of all bodies of the mechanical system [88]. There are several ways to solve this problem and different authors have opted for one way or another depending on their preferences. There are two types of coordinate systems which can be adopted during the process of modeling of a multi-body mechanical system: The first type is that of using a system of *Independent Coordinates* [1], [88]–[90] whose number coincides with the number of degrees of freedom of the multi-body system, while the second type is the use of *Dependent Coordinates* [1], [88]–[90] whose number is greater than the number of degrees of freedom of the multi-body system.

Studies have shown that generally a system of Independent Coordinates is not an acceptable solution, since they directly determine the position of the input links but not the position of the other links, hence they cannot be used to define unequivocally at all times the position, velocity and acceleration of the entire multi-body system [88]. Dependent Coordinates are more preferred because they describe the multi-body system more easily, and are related by the algebraic constraint equations which are generally non-linear and play a vital role in the kinematics and dynamics of multi-body system. The number of these algebraic constraint equations relating the dependent coordinates, is equal to the difference between the number of selected dependent coordinates and the number of degrees of freedom of the multi-body system [90].

Dependent Coordinates are further categorized as, *Relative Coordinates* and *Cartesian Coordinates*.

(a) Relative Coordinates : These define the position of each link in relation to the previous link in the kinematic chain by using a parameter which corresponds to the relative degree of freedom allowed by the joint connecting the links. In case of a planar multi-body system, if two links are connected by a revolute joint the coordinate defining their relative position is an angle, while if the two links are connected by a prismatic joint, their relative position is defined by means of a distance. Relative coordinates lead to a system with a minimum number of dependent coordinates and are suited for open-loop multi-body systems since the number of relative coordinates for a open-loop configuration is equal to the number of degrees of freedom, and hence there will be no required constraint equations. Although the relative coordinates lead to reduced number of dependent coordinates, they result to equations of motion with small but full matrices which are computationally intensive to evaluate. Due to this, relative coordinates are not widely used as compared to cartesian coordinates.

(b) Cartesian Coordinates : This set of dependent coordinates try to overcome the limitations of relative coordinates by directly defining using three coordinates, the absolute position of each link in the multi-body system. The three coordinates for each link are normally obtained by determining the position of a point on the link (preferably the center of gravity) with two x and y linear coordinates, and also determining the angular orientation of the body relative to a global inertial axis. Although the cartesian coordinates lead to a large number of dependent coordinates as compared to relative coordinates, they result to equations of motion with large but sparse matrices (matrices with very few non-zero elements) which can be used to make the formulation numerically efficient. Garcia de Jalon and Bayo [88] modified these cartesian coordinates by moving the reference points to the joints so that each link has two points defined with cartesian coordinates. This set of coordinate system is called Natural Cartesian Coordinate and leads to a decrease in the number of dependent coordinates due to the elimination of angular coordinates, and the sharing of coordinates by two links at the joint connecting them. Although the natural cartesian coordinates lead to slightly lower dependent coordinates and hence computationally effective formulation as compared to absolute cartesian coordinates, computational implementation is not as easy as the use of absolute cartesian coordinates [78]

For further and more information on comparison of cartesian and relative coordinate systems, any interested leader is advised to refer to [88]. Due to its simplicity and ease of computational implementation, Absolute Cartesian Coordinates are widely employed in formulating the equation of motion of multi-body mechanical system.

B. Computational Kinematic Analysis of Multi-body Systems

1) Generalized Coordinate Vector: The translational motion of a rigid body i can be defined by the vector $\mathbf{R_i}$ that describes the position of the origin of the body reference coordinate system with respect to the global coordinate system, while the orientation of the body with respect to global coordinate system can be described using the angle θ_i as shown in Figure 2.

Therefore, the vector $\mathbf{q_i}$ of the generalized absolute cartesian coordinates of body i in a multi-body system can be represented as,

$$\mathbf{q_i} \;=\; \Big[\mathbf{R_i} \;\; \theta_i\Big]^T$$

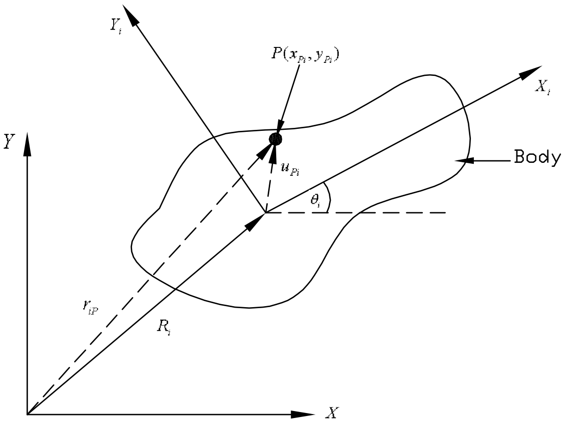

Fig. 2.

$$= \Big[R_{ix} \;\; R_{iy} \;\; \theta_i\Big]^T$$

XY is the global coordinate system, X_iY_i is the coordinate system of body i, $\mathbf{u_{Pi}}$ is the position vector of point P on body i with respect to the X_iY_i coordinate system, $\mathbf{R_i}$ is the position vector of the origin of X_iY_i coordinate system with respect to the the global XY coordinate system, $\mathbf{r_{iP}}$ is the position vector of point P on body i with respect to the global XY coordinate system, and θ_i is the angle in which the X_iY_i coordinate system is oriented relative to the the global XY coordinate system.

The global position vector of an arbitrary point P on body i can be represented as;

$$\begin{aligned} \mathbf{r_{iP}} &= \mathbf{R_i} + \mathbf{u_{Pi}} \\ &= \mathbf{R_i} + A_i\overline{u}_{Pi} \end{aligned} \qquad (12)$$

Where A_i is the transformation matrix from the global XY coordinate system to the X_iY_i coordinate system, given as;

$$A_i \;=\; \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix}$$

and $\overline{u}_{Pi}$ is the coordinates of point P on body i measured about the coordinate system of body i, and given as;

$$\overline{u}_{Pi} \;=\; \begin{pmatrix} x_{Pi} \\ y_{Pi} \end{pmatrix}$$

A multi-body system consisting of n_b rigid bodies, has $3 \times n_b$ absolute cartesian coordinates. Hence a vector $\mathbf{q}$ of the generalized coordinates of a multi-body system is defined as;

$$\mathbf{q} \;=\; \Big[R_{1x} \;\; R_{1y} \;\; \theta_1 \;\; R_{2x} \;\; R_{2y} \;\; \theta_2 .... R_{n_bx} \;\; R_{n_by} \;\; \theta_{n_b}\Big]^T$$

which can be written as,

$$\mathbf{q} \;=\; \Big[\mathbf{q_1} \;\; \mathbf{q_2} \;\; \mathbf{q_3} \cdots\cdots\cdots \mathbf{q_{n_b}}\Big]^T$$

where $\mathbf{q_i}$ is the vector of generalized cartesian coordinates of body i, given as;

$$\mathbf{q_i} \;=\; \Big[R_{ix} \;\; R_{iy} \;\; \theta_i\Big]$$

Kinematic constraints impose restrictions on the relative motion between bodies, and are classified as either joint kinematic constraints or driving constraints.

2) Formulation of Joint Constraints: The algebraic joint constraint equations are formulated in terms of the absolute cartesian coordinates that describe the position and orientation of the rigid bodies with respect to a fixed global coordinate system.

Formulation of Ground Constraints: A body that has zero degrees of freedom is called ground or fixed link. If *i* is a ground link, the algebraic kinematic constraints are given as,

$$\begin{aligned} R_{ix} - C_1 &= 0 \\ R_{iy} - C_2 &= 0 \\ \theta_i - C_3 &= 0 \end{aligned}$$

where C_1, C_2 and C_3 are constants. These constraints eliminate the translation and rotational freedoms of the body. These constraint equations can also be written as,

$$\mathbf{q_i} - \mathbf{C} = 0$$

where

$$\mathbf{q_i} = \begin{pmatrix} R_{ix} \\ R_{iy} \\ \theta_i \end{pmatrix}$$
$$\mathbf{C} = \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix}$$

If the global coordinate system coincides with the coordinate system of fixed link *i*, then all the constants are equal to zero.

Formulation of Revolute Joint Constraints: As shown in Figure 4, bodies *i* and *j* are connected by a revolute joint at point P called the joint definition point.

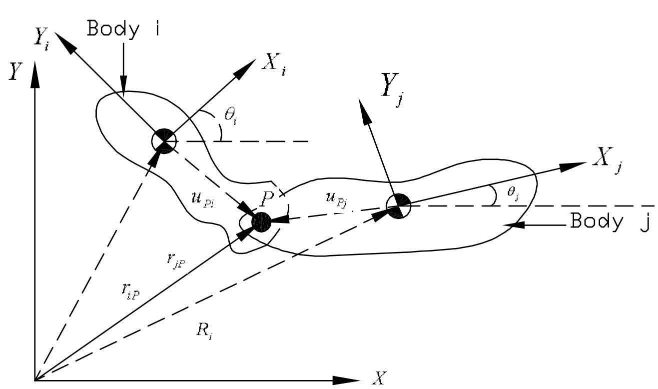

Fig. 3. Planar revolute joint connecting bodies *i* and *j*

Throughout the motion, the revolute joint requires that point P on body *i* (P_i) always remains in contact with point P on body *j* (P_j). Mathematically, this kinematic constraint condition for the revolute joint can be expressed as,

$$\begin{aligned} \mathbf{r_{iP}} &= \mathbf{r_{jP}} \\ \mathbf{R_i} + A_i \overline{u}_{Pi} &= \mathbf{R_j} + A_j \overline{u}_{Pj} \\ \mathbf{R_i} + A_i \overline{u}_{Pi} - \mathbf{R_j} - A_j \overline{u}_{Pj} &= 0 \end{aligned} \quad (13)$$

where $\overline{u}_{Pi} = [x_{Pi} \;\; y_{Pi}]^T$ and $\overline{u}_{Pj} = [x_{Pj} \;\; y_{Pj}]^T$ are the local position vectors of point P defined with respect to the coordinate systems of body *i* and body *j* respectively.

Equation 13 can be written explicitly as,

$$0 = \begin{pmatrix} R_{ix} \\ R_{iy} \end{pmatrix} + \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix} \begin{pmatrix} x_{Pi} \\ y_{Pi} \end{pmatrix} - \begin{pmatrix} R_{jx} \\ R_{jy} \end{pmatrix} - \begin{pmatrix} \cos\theta_j & -\sin\theta_j \\ \sin\theta_j & \cos\theta_j \end{pmatrix} \begin{pmatrix} x_{Pj} \\ y_{Pj} \end{pmatrix}$$

which when evaluated gives two scalar equations that eliminate the freedom of the bodies to translate relative to one another.

Formulation of Prismatic Joint Constraints: A prismatic joint removes two degrees of freedom from a planar multibody system, that is, there is neither rotation motion between the bodies nor a relative translation motion in the direction perpendicular to the joint axis. Therefore, a planar prismatic joint is represented by two independent algebraic constraint equations.

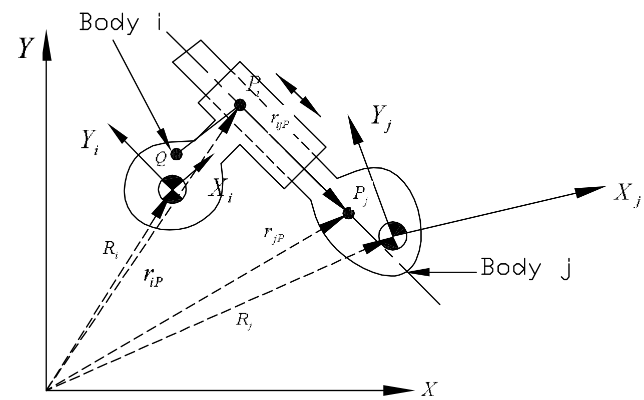

Fig. 4. Planar prismatic joint connecting bodies *i* and *j*

A constraint equation that eliminates the relative rotation between bodies *i* and *j* can be written as,

$$\theta_i - \theta_j - C = 0 \quad (14)$$

C is a constant defined by $C = \theta_{io} - \theta_{jo}$ where θ_{io} and θ_{jo} are the initial orientation angles for body *i* and *j* respectively. Let

(a) r_{ijP} be a position vector connecting two arbitrary points P_i and P_j on bodies *i* and *j* respectively, and lying on the axis of the joint.

(b) h_i be a vector perpendicular to the joint axis and defined on body *i*. This vector is selected to join point P_i and an arbitrary point Q on body *i*.

In order to eliminate the relative translation between the two bodies *i* and *j* along the axis perpendicular to the joint axis, the vectors r_{ijP} and h_i must always remain perpendicular, and hence their dot product must be zero.

$$\begin{aligned} h_i . r_{ijP} &= 0 \quad (15) \\ r_{ijP} &= R_i + A_i \bar{u}_{Pi} - R_j - A_j \bar{u}_{Pj} \\ h_i &= A_i(\bar{u}_{Pi} - \bar{u}_{Qi}) \end{aligned}$$

Equation 14 and 15 can be combined to yield two constraint equations for a planar translational joint as,

$$\begin{pmatrix} \theta_i - \theta_j - (\theta_{io} - \theta_{jo}) \\ \left[A_i(\bar{u}_{Pi} - \bar{u}_{Qi})\right] . \left[R_i + A_i \bar{u}_{Pi} - R_i - A_j \bar{u}_{Pj}\right] \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (16)$$

The first equation in 16 is a linear function of the rotational coordinates of body *i* and body *j*, while the second equation is a nonlinear equation of the absolute coordinates of the two bodies.

Formulation of Driving Constraints: Unlike the joint constraints which depend solely on the system coordinates, driving constraints describe the specified motion trajectories and therefore may depend on the system generalized coordinates as well as time. The maximum number of driving constraints imposed on the motion of a given multi-body system must equal the number of degrees of freedom of the system.

3) Position Analysis: For kinematically driven systems, the total number of algebraic constraint equations (n_c) must be equal to the number of generalized coordinate systems (n). Hence the constraint equations vector contains $n = n_c$ algebraic equations that describe the joint constraints and driving constraints. Since the driving constraints depend also with time, the constraint equations vector can be written as,

$$C(\mathbf{q},t) \;=\; \Big[C_1(\mathbf{q},t)\;\; C_2(\mathbf{q},t)\;\; ....C_n(\mathbf{q},t)\Big]^T = 0$$

This equation contains n nonlinear scalar equations which can be solved for the n unknown generalized coordinates given in a vector form as,

$$\mathbf{q} \;=\; \Big[\mathbf{q_1}\;\; \mathbf{q_2}\;\; \mathbf{q_3}\;\; ....\mathbf{q_n}\Big]^T$$

There are numerical procedures used to solve a system of nonlinear algebraic equations, but the mostly preferred algorithm is the Newton-Raphson algorithm which involves linearizing the set of nonlinear equations of kinematic constraints to get the first order approximation of equation $C(\mathbf{q},t) = 0$ as,

$$C_{q_i}\Delta q_i \;=\; -C(\mathbf{q}_i,t) \tag{17}$$

where,

(a) C_{q_i} is the Jacobian matrix at iteration point *i*, given as,

$$C_{q_i} \;=\; \begin{pmatrix} \frac{\partial C_1}{\partial q_1} & \frac{\partial C_1}{\partial q_2} & \frac{\partial C_1}{\partial q_3}\cdots\cdots & \frac{\partial C_1}{\partial q_n} \\ \frac{\partial C_2}{\partial q_1} & \frac{\partial C_2}{\partial q_2} & \frac{\partial C_2}{\partial q_3}\cdots\cdots & \frac{\partial C_2}{\partial q_n} \\ \frac{\partial C_3}{\partial q_1} & \frac{\partial C_3}{\partial q_2} & \frac{\partial C_3}{\partial q_3}\cdots\cdots & \frac{\partial C_3}{\partial q_n} \\ . & . & . & . \\ . & . & . & . \\ \frac{\partial C_n}{\partial q_1} & \frac{\partial C_n}{\partial q_2} & \frac{\partial C_n}{\partial q_3}\cdots\cdots & \frac{\partial C_n}{\partial q_n} \end{pmatrix} \tag{18}$$

For kinematically driven system, the Jacobian matrix is a square non-singular matrix.

(b) Δq_i is the vector of Newton differences at iteration point *i*

(c) $C(\mathbf{q}_i,t)$ is the vector of constraint equations at iteration point *i*.

Since the jacobian matrix is non-singular, the vector of Newton differences Δq_i can be solved from equation 17. This vector can be used to iteratively update the vector of the systems coordinates as $q_{i+1} = q_i + \Delta q_i$. The updated vector q_{i+1} is then used to reconstruct equation 17 to solve for Δq_{i+1} which can then be used again to update the vector of the system coordinates $q_{i+2} = q_{i+1} + \Delta q_{i+1}$. This iteration process continues until the solution converges provided that the assumed initial solution for the system coordinates was chosen to be near the actual solution.

4) Velocity Analysis: Differentiating the vector of constraint equations with respect to time yields a vector of velocity constraint equations as,

$$C_q\dot{q} \;=\; -C_t \tag{19}$$

where C_q is the constraint Jacobian matrix given by equation 18 and C_t is the vector of partial derivatives of the constraint equations with respect to time, which is given as,

$$C_t \;=\; \Big[\; \frac{\partial C_1}{dt}\;\; \frac{\partial C_2}{dt}\;\; \frac{\partial C_3}{dt}\cdots\cdots\;\; \frac{\partial C_n}{dt} \;\Big]^T \tag{20}$$

If all the constraint equations are not explicit functions of time, then the vector C_t is identically a zero vector. Equation 19 is a linear system of algebraic equations in terms of $\dot{q}$ and can be solved easily for $\dot{q}$ by any usual methods for systems of linear equations.

5) Acceleration Analysis: Differentiating the vector of velocity constraint equations given in equation 19 yields a vector of acceleration constraint equations as,

$$C_q\ddot{q} \;=\; -(C_q\dot{q})_q\dot{q} - 2C_{qt}\dot{q} - C_{tt} \tag{21}$$

where

1) C_{qt} is the time derivative of the Jacobian matrix given as,

$$C_{qt}=\begin{pmatrix} \frac{\partial}{\partial t}\Big[\frac{\partial C_1}{\partial q_1}\Big] & \frac{\partial}{\partial t}\Big[\frac{\partial C_1}{\partial q_2}\Big] & \frac{\partial}{\partial t}\Big[\frac{\partial C_1}{\partial q_3}\Big]\cdots\cdots & \frac{\partial}{\partial t}\Big[\frac{\partial C_1}{\partial q_n}\Big] \\ \frac{\partial}{\partial t}\Big[\frac{\partial C_2}{\partial q_1}\Big] & \frac{\partial}{\partial t}\Big[\frac{\partial C_2}{\partial q_2}\Big] & \frac{\partial}{\partial t}\Big[\frac{\partial C_2}{\partial q_3}\Big]\cdots\cdots & \frac{\partial}{\partial t}\Big[\frac{\partial C_2}{\partial q_n}\Big] \\ \frac{\partial}{\partial t}\Big[\frac{\partial C_3}{\partial q_1}\Big] & \frac{\partial}{\partial t}\Big[\frac{\partial C_3}{\partial q_2}\Big] & \frac{\partial}{\partial t}\Big[\frac{\partial C_3}{\partial q_3}\Big]\cdots\cdots & \frac{\partial}{\partial t}\Big[\frac{\partial C_3}{\partial q_n}\Big] \\ . & . & . & . \\ . & . & . & . \\ \frac{\partial}{\partial t}\Big[\frac{\partial C_n}{\partial q_1}\Big] & \frac{\partial}{\partial t}\Big[\frac{\partial C_n}{\partial q_2}\Big] & \frac{\partial}{\partial t}\Big[\frac{\partial C_n}{\partial q_3}\Big]\cdots\cdots & \frac{\partial}{\partial t}\Big[\frac{\partial C_n}{\partial q_n}\Big] \end{pmatrix} \tag{22}$$

2) C_{tt} is the vector of second partial derivatives of the constraint equations with respect to time, which is given as,

$$C_{tt} \;=\; \Big[\; \frac{\partial^2 C_1}{dt^2}\;\; \frac{\partial^2 C_2}{dt^2}\;\; \frac{\partial^2 C_3}{dt^2}\cdots\cdots\;\; \frac{\partial^2 C_n}{dt^2} \;\Big]^T \tag{23}$$

Equation 21 is a linear system of algebraic equations in terms of $\ddot{q}$ and can be solved easily for $\ddot{q}$ by any usual methods for systems of linear equations.

Therefore, computational kinematic analysis of a multi-body system is performed by solving the set of equations 17, 19 and 21 for q, $\dot{q}$ and $\ddot{q}$ respectively. This formulation can easily be implemented on a digital computer, and made available for kinematic analysis of a large class of multi-body mechanical systems.

C. Computational Dynamic Analysis of Multi-body Systems

Generally, the equation of motion of a multi-body system can be written as,

$$M\ddot{q} \;=\; Q_e + Q_c \tag{24}$$

where M is the mass matrix of the system, $\ddot{q}$ is the vector of the system acceleration, Q_e is a vector containing the external forces which are known and Q_c is a vector of the reaction forces which are unknown and should be calculated.

The joint reaction forces can be expressed in terms of the Jacobian matrix of the constraint equations and a vector of Lagrange multipliers as,

$$Q_c = -C_q^T \lambda$$

where λ is the vector that contains m unknown Lagrange multipliers. Hence equation 24 becomes,

$$M\ddot{q} + C_q^T \lambda = Q_e \tag{25}$$

Equation 25 represents n scalar equations but with $n + m$ unknowns. In order to have a sufficient number of equations, it becomes necessary to supply m more equations. This is achieved by considering the algebraic constraint equations simultaneously with the differential equations of motion. Differentiating the vector of algebraic constraint equations twice with respect to time and arranging the terms so as to isolate the generalized acceleration of coordinates, the acceleration constraints are gotten as represented in equation 21. Therefore equation 21 is combined with equation 25 to yield a system of Differential Algebraic Equation (DAE) which can be represented as,

$$\begin{pmatrix} M & C_q^T \\ C_q & 0 \end{pmatrix} \begin{pmatrix} \ddot{q} \\ \lambda \end{pmatrix} = \begin{pmatrix} Q_e \\ -(C_q\dot{q})_q\dot{q} - 2C_{qt}\dot{q} - C_{tt} \end{pmatrix} \tag{26}$$

If we let $-(C_q\dot{q})_q\dot{q} - 2C_{qt}\dot{q} - C_{tt} = \gamma$ then, equation 26 can be written as,

$$\begin{pmatrix} M & C_q^T \\ C_q & 0 \end{pmatrix} \begin{pmatrix} \ddot{q} \\ \lambda \end{pmatrix} = \begin{pmatrix} Q_e \\ \gamma \end{pmatrix} \tag{27}$$

Therefore, equation 27 shows that, mathematically the simulation of a constrained multi-body system requires the solution of n differential equations coupled with a set of m algebraic equation. Dynamic analysis of a multi-body mechanical system involves entirely the solving of acceleration vector $\ddot{q}$ and the Lagrange multipliers vector λ. Once the generalized acceleration vector $\ddot{q}$ is obtained, it can be integrated once and twice to get the generalized velocity vector $\dot{q}$ and the generalized position vector q of the multi-body system respectively.

The differential algebraic equation 27 can be solved for $\ddot{q}$ and λ either directly or inversely. In inverse solution, an expression for the accelerations is first obtained from equation 25 as,

$$\ddot{q} = M^{-1}Q_e - M^{-1}C_q^T \lambda$$

which can be substituted back in equation 21 to solve for Lagrange multipliers.

If the initial vectors q and $\dot{q}$ are known, then equation 27 can be solved for $\ddot{q}$ and λ by direct integration. In majority of practical cases, the direct solution is preferred since λ is obtained without the need for special call to an inverse dynamic module. However the direct solution of such type of equations and their integration with time introduces several numerical difficulties, namely, the existence of an uniqueness of solutions and instability for higher order systems [88]. Special numerical algorithm which use multi-stepping procedures and have the ability to deal with stiff systems are often required to solve such a set of differential algebraic equations directly [91], [92].

An alternative and preferred approach for the direct solution of the equations of motion given in equation 27 is to transform the set of DAE to its governing set of ordinary differential equations (ODE) which are then solved by integration with time. However, the substitution of the algebraic equations of the DAE system by their ODE counterparts introduces mild instabilities and drift problems during the integration process. The drift phenomenon means that for long simulations, the original constraint equations for position and velocity begin to be violated during the integration process. This happens because the equations of motion represented by differential algebraic equation 27 does not use explicitly the position and velocity equations associated with the kinematic constraints. Therefore, special procedures must be adopted to avoid or minimize this drift phenomenon. Several methods to solve this problem have been suggested and tested, of which the most common are; Baumgarte Stabilization Method (BSM) [93], the Coordinate Partitioning Method (CPM) [94] and the Augmented Lagrangian Formulation (ALF) [95].

In CPM, the generalized coordinates are partitioned into independent and dependent sets of coordinates. The numerical integration is carried out for the independent generalized coordinates. Then, the constraint equations are solved for the dependent generalized coordinates using Newton-Raphson method. The advantage of this method is that it satisfies all the constraints to the level of precision specified and maintains good error control. However, it suffers from poor numerical efficiency due to the requirement for the iterative solution for dependent generalized coordinates in the Newton-Raphson method [88], [89], [94].

ALF is based on Hamiltons principle and the constraint equations are taken into account using a penalty approach. This method involves solving the system's equations of motion, represented by Equation 27 using an iterative process. The form of the constraint equations is similar to the form proposed in the Baumgartes method but it has the advantage of handling redundant constraints in the process [88], [95]

Due to simplicity and easiness for computational implementation, the Baumgarte Stabilization Method is the most preferred technique to overcome the drawbacks of the direct integration of the equations of motion of a multi-body system. Although BSM gives good results in most of the applications, it does not help in the cases of some particular configuration of the systems, such as near kinematic singularities [88]. The aim of the BSM is to replace the differential equation 21 by the following equation [93],

$$\ddot{C} + 2\alpha\dot{C} + \beta^2 C = 0$$

which is a differential equation for a closed loop system in terms of kinematic constraint equation $C(q, t)$. The terms $2\alpha\dot{C}$ and $\beta^2 C$ play the role of feedback control, and the parameters α and β are termed as feedback parameters.

The principle of BSM is to damp out the acceleration constraint violations by feeding back the violations of the position and velocity constraints. Thus, by using the Baumgarte's approach, the equations of motion for a dynamic system subjected to holonomic constraints are represented as,

$$\begin{pmatrix} M & C_q^T \\ C_q & 0 \end{pmatrix} \begin{pmatrix} \ddot{q} \\ \lambda \end{pmatrix} = \begin{pmatrix} Q_e \\ \gamma - 2\alpha\dot{C} - \beta^2 C \end{pmatrix} \quad (28)$$

The challenge in Baumgarte's method is the criterion for choosing the adequate values of α and β. Initially, Baumgarte [93] pointed out that the stabilizing values of $\alpha = \beta = 5$ are a good choice for a multi-body system made up rigid bodies, however a lot of research work has been done in order to modify the first principle Baumgarte method and also to come up with criteria for choosing optimal values of the feedback parameters [96]–[99]

There is no precise method for choosing the most correct values for the feedback parameters for general cases. The choice of α and β usually involves the use of trial-and-error-procedure, but these feedback parameters should be equal to one another, and the their typical values range from 1 to 20 [88], [97]

IV. Conclusion

A thorough literature on dynamic analysis of multi-body mechanical systems when imperfections (imperfect kinematic joints and the link flexibility) are considered has been critically reviewed. From the carried review it has been observed that, the serious consequences of the imperfect kinematic joints on the dynamic response of the mechanical system have attracted many theoretical and experimental investigations over the last three decades. However:

(a) Many of the tribological effects at the joint, such as the impact forces, friction, lubrication and wear have been studied either individually or in some rare instances considering a combination of few effects, and thus very few formulations present comprehensive models for predicting the dynamic response of multi-body mechanical systems with imperfect joints.

(b) A lot of emphasis has been laid on rigid multi-body mechanical systems. In the current efforts to search for higher efficiency, low cost and greater productivity in machines, links are currently being designed to be of lighter weight, to carry larger loads, and to operate at higher speeds. The light members are more flexible and hence the standard rigid body mechanism models can no longer be used to accurately predict the behavior of these systems. Therefore, in order to fully exploit the potential offered by flexible links, the effect of flexibility must be accounted for in a dynamic model. This demands for a comprehensive study on flexible multi-body dynamics.

(c) Many researchers have modeled only one joint in the considered mechanisms as a real (imperfect) joint while assuming the other joints to be ideal (perfect). Although, the results from such experimental and analytical models have been shown to provide important insights on the behavior of mechanical systems with imperfect joints, the models do not allow for study of the interactions of multiple kinematic imperfect joints. Furthermore a real mechanical system does not have only one real joint, but practically all joints are real. This led several researchers such as Flores (2004) [78] and Cheriyan (2006) [100] to strongly recommend for their work to be extended to include multi-body mechanical systems with multiple imperfect joints, and with a variety of joints such as prismatic and universal joints. Few recent papers by Erkaya and Uzmay (2009-2010) [24], [30] have considered the nonlinear dynamic analysis of multi-body systems with two imperfect joints. However in these research papers, only mechanisms with rigid links have been considered and the interaction effects of the imperfect joints on the overall response of a multi-body system were not investigated.

(d) The dry friction at the joint has been widely modeled by modifying the classical Coulomb friction law to avoid the discontinuity of force at zero relative velocity and to obtain a continuous frictional force-velocity relationship. Although this modification of Coulomb's law allows for the numerical stabilization of the integration algorithm, it does not account for the stiction phenomena of the contacting surfaces which occurs when the relative tangential velocity of the two impacting bodies approaches zero. This led Flores (2004) [78] in his PhD Dissertation to strongly recommend for his work to be extended to model and include stick-slip friction at imperfect joints of a multi-body mechanical system.

(e) Although dynamic modeling work for the flexible linkage multi-body systems has attracted more attention and gained some achievements, nonlinear dynamic analysis caused by multiple imperfect joints has not been carried out more widely.

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