WISHART STOCHASTIC VOLATILITY MODELS WITH APPLICATIONS TO EMERGING FINANCIAL MARKETS DATA

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Declaration

I, Nabirye Topilista (MF300-0009/15) declare that this research thesis is my original work which has never been submitted for any other degree award in any other university before.

Signature .... Date ....

Supervisor's Declaration

We approve that the information presented in this research thesis was as a result of the work carried out by Nabirye Topilista under our supervision.

1. Signature .... Date ....

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   Jomo Kenyatta University of Agriculture and Technology

Dedication

I dedicate this work to my dear husband Mr. Joseph Okello Omwonylee, my parents Mr and Mrs Balongoire William, my children Brenda, Francis and Agatha, my sisters and my brothers.

Acknowledgements

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$M_{n,m}(\mathbb{R})$ - Set of real matrices $n \times m$

$\sigma(M)$ - Set of the eigenvalues of the matrix M

$GL_n(\mathbb{R})$ - Set of real invertible matrices $n \times n$

$S_n(\mathbb{R})$ - Set of real symmetric matrices

$S_n^+(R)$ - Set of nonnegative symmetric matrices

A' - Transpose matrix of A

$Tr(A)$ - Trace of the matrix A

Abbreviations

ATS - Affine Term Structure

CIR - Cox - Ingersoll - Ross

ODF - Ordindary Differential Equation

ADF - Augumented Dickey Fuller

LM - Lagrange Multiplier

AIC - Akaike Information Criterion

BIC - Bayesian Information Criterion

Abstract

Financial markets are known to be far from deterministic but stochastic and hence random models tend to perfectly model the markets. The most recent development in stochastic models is the Wishart Stochastic Volatility Model which is a n dimensional model. The study aimed at modelling returns volatility in emerging financial market using Wishart Stochastic Volatility Model. Pricing in one dimension and two dimension was explored. A suitable Wishart Stochastic Volatility Model for an emerging financial market was constructed basing on the characteristics of an emerging financial market. Foreign Exchange derivative pricing was done under constant and stochastic correlation using finite difference method called the Crank Nicolson method. The study compared the modified Model (with stochastic correlation) to the Black scholes model (with constant correlation) using real data from emerging financial markets that is the exchange rates data for Kenya as the domestic currency and South Africa as the foreign currency. The modified model provide better volatility smiles compared to the Black scholes model and outperformed the Black scholes model as observed from the smallest AIC and BIC values.

Chapter 1

Introduction

1.1 Background

The Wishart process was originally studied by Bru (1991) and was introduced in finance by Gourieroux in 2004. Since it was introduced in finance, different authors have developed stochastic volatility models using it and this paper focuses on the Wishart Stochastic Volatility Model presented by Da Fonseca and others. Due to the weakness of Black-Scholes model of not incorporating the observable phenomenon that implied volatility of derivative products is strike and maturity dependent, various models have been introduced such as local volatility and stochastic volatility models to reproduce some market conditions. The first step was the introduction of local volatility models by Dupire (1994) where the underlying volatility $\sigma(t, S)$ depends on the level of the underlying S itself. The most famous one is the Constant Elasticity of Variance model (CEV) in which the volatility is proportional to S^α, where α is a positive constant. Stochastic volatility models appear where volatility is assumed to be a stochastic process. Thus, models became more complex because of the market incompleteness which implies that traders cannot hedge their products by dealing only with the asset. The model by Heston (1993) presents a volatility with an effect of mean reversion,

and is commonly used in financial markets because of its flexibility. However, these models cannot fit accurately market data for short or long maturities, and recent research have been carried out to improve this point. A way to solve this problem was the introduction of a multifactor stochastic volatility model. Gouriéroux (2006) developed a multifactor version of the Heston model. It was assumed that volatility follows a Wishart process introduced in 1991 so that the model preserves its linear properties and consequently its tractability. da Fonseca, Grasselli, and Tebaldi (2005) have improved the model by considering that the volatility of the asset is the trace of a Wishart process. This allows to take into account stochastic correlation between the underlying asset with the volatility process and provides a wealthy but complex model. The simplistic case where the matrix of mean reversion, the volatility of volatility matrix and the correlation matrix are diagonal matrices gives a small intuition of the model performance given that the diagonal components of the Wishart process are in fact Cox-Ingersoll-Ross processes : by considering the volatility as the trace of the Wishart process, the model is equivalent to a simple multifactor Heston model.

However, the multifactor Heston model is not flexible enough in regard to the stochastic correlation. Indeed, in a classic extension of a multifactor Heston model, the factors appearing in the stochastic correlation formulae are exactly the same as the volatility expression's ones. The model is extended by focusing on a specification of the Wishart Stochastic Volatility Model of allowing to add freedom degrees concerning the stochastic correlation.

Given a filtered probability space $(\Omega, F, F_t, \mathbb{P})$ and a $n \times n$ matrix Brownian motion W (i.e. a matrix whose entries are independent Brownian motions under $\mathbb{P}$), a Wishart process on Σ_n^+ is governed by the Stochastic differential equation (The matrix-valued process Σ is said to be a Wishart process if it satisfies the following SDE)

$$d\Sigma(t) = (KQ'Q + M\Sigma(t) + \Sigma(t)M')dt + \sqrt{\Sigma(t)}dW(t)Q + Q'dW(t)'\sqrt{\Sigma(t)} \quad (1.1)$$

where $\Sigma(0) = c_0 \in \Sigma_n^+$, $t \geq 0$, $Q \in GL_n(\mathbb{R})$, $M \in M_n$ (the set of real $n \times n$ matrices) with all eigenvalues on the negative real line in order to ensure stationarity, and where the (Gindikin) real parameter $K > n - 1$. The condition $K > n - 1$ is introduced to ensure existence and unicity of the solution $\Sigma_t \in \Sigma_n^+$ of equation (1.1).

1.2 Statement of the problem

Due to the importance of the study on volatility, there has been numerous research and various approaches on it see Ahmed and Suliman (2011), Abdalla and Winker (2012), Koleva and Nicolato (2012) and others. Among them, the two main approaches are deterministic models and stochastic models. Deterministic models assume that volatility at a particular time follows a deterministic function of the past, She (2013) whereas stochastic models assume that the volatility follows certain random process, Eisler, Perelló, and Masoliver (2007). Different authors have thoroughly modeled volatility using different deterministic models but little attention has been given to the stochastic volatility models due to their complexity especially to empiricists. However, financial markets are known to be far from deterministic and hence random models tend to perfectly model the market. The most recent development in stochastic models is the Wishart Stochastic Volatility Model. Notwithstanding, there has not been a comprehensive study on the suitability of Wishart Stochastic Volatility Model to emerging financial markets.

1.3 Objectives of the research

1.3.1 Main objective

To model volatility in the emerging financial markets using Wishart Stochastic Volatility Models.

1.3.2   Specific objectives

The specific objectives of this study are:

1. To construct a Wishart Stochastic Volatility Model for emerging financial markets.

2. To apply the newly constructed Wishart Stochastic Volatility Model in pricing of European call options.

3. To compare results of a newly constructed Wishart Stochastic Volatility Model to the Black-Scholes model using data from emerging markets.

1.4   Significance of study

Volatility refers to the measure for price fluctuation of specific financial instrument over time. It is a very important factor that can greatly influence investors decisions and concerns every other participant in the financial markets. The higher the volatility, the riskier the security which means that volatility is related to risks (volatility is used as a measurement of risk).

Many financial time series exhibit volatility clustering which means that the series have periods where volatility is low and other periods where volatility is high. Due to usefulness of volatility, various models have been developed since the introduction of the Autoregressive Conditional Heteroscedasticity (ARCH) model by Engle (1982) especially for data that shows volatility clustering that is the deterministic models and stochastic models. In emerging financial markets, the price movements affect people making it important to model volatility of returns using the stochastic models which assumes that volatility follows certain random process. The study will help in understanding the suitability of Wishart Stochastic Volatility Model to emerging financial markets.

Chapter 2

Literature Review

The literature review is presented in two aspects: Volatility analysis and Wishart process. In volatility analysis, the existing literatures on both deterministic volatility models and stochastic volatility models are discussed. And in the section of wishart process, the existing literatures on wishart process are discussed.

2.1 Volatility Analysis

Engle (1982) introduced the Autoregressive Conditional Heteroscedasticity (ARCH) model to model volatility by relating the conditional variance of the disturbance term to the linear combination of the squared disturbances in the recent past. Bollerslev (1986) generalized the ARCH model by modeling the conditional variance to depend on its lagged values as well as squared lagged values of disturbance. Since the works of Engle (1982) and Bollerslev (1986), various variants of GARCH model have been developed to model volatility. Some of the models include EGARCH originally proposed by Nelson (1991), GJR-GARCH model introduced by Glosten, Jagannathan, and Runkle (1993), Threshold GARCH (TGARCH) model by Zakoian (1994) among others. Following the success of the ARCH family models in capturing behaviour of volatility, Stock returns volatility has received a great attention from both academies

and practitioners as a measure and control of risk both in emerging and developed financial Markets. Although the GARCH model has been very successful in capturing important aspect of financial data, particularly the symmetric effects of volatility, it has had far less success in capturing extreme observations and skewness in stock return series. The Traditional Portfolio Theory assumes that the (logarithmic) stock returns are independent and identically distributed (iid) normal variables which do not exhibit moment dependencies, but a vast amount of empirical evidence suggest that the frequency of large magnitude events seems much greater than is predicted by the normal distribution.

Kama, Haq, Ghani, and Khan (2012) modelled volatility of the Pakistani Rupee and US Dollar using the GARCH in mean model, EGARCH and TARCH models. According to their results, it was concluded that EGARCH model was the best model in explaining the volatility behaviour of the exchange rate of Pakistan against the US Dollar. However financial markets are far from deterministic but stochastic.

Stochastic volatility models are those in which the volatility follows a random process. They are used in the field of mathematical finance to evaluate derivative securities, such as options. The name is derived from the models' treatment of the underlying security's volatility as a random process, governed by state variables such as the price level of the underlying security, the tendency of volatility to revert to some long-run mean value, and the variance of the volatility process itself, among others. Stochastic volatility models are important since they resolve the shortcomings of the Black-Scholes model. In particular, models based on Black-Scholes assume that the underlying volatility is constant over the life of the derivative, and unaffected by the changes in the price level of the underlying security. However, these models cannot explain long-observed features of the implied volatility surface such as volatility smile and

skew, which indicate that implied volatility tend to vary with respect to strike price and expiry. By assuming that the volatility of the underlying price is a stochastic process rather than a constant, it becomes possible to model derivatives more accurately. The stochastic volatility models family were proposed by several different researchers via various models for example Ahn and Wilmott (2005), Philipov and Glickman (2006b), Gouriéroux, Jasiak, and Sufana (2009), Asai, McAleer, and Yu (2006), Philipov and Glickman (2006a), among others.

Among the stochastic models, one kind of process called the Wishart process with different construction has been proposed by several independent researchers. The basic assumption of Wishart process is that returns follow a multivariate distribution, while the covariance matrix obey a one order autoregressive Markov chain.

Philipov and Glickman (2006), proposed a high dimensional factor multivariate stochastic volatility (MSV) model in which factor covariance matrices are driven by Wishart random processes. Their framework allows for unrestricted specification of inter temporal sensitivities, which can capture the persistence in volatilities, kurtosis in returns, and correlation breakdowns and contagion effects in volatilities. The factor structure allows addressing high dimensional setups used in portfolio analysis and risk management, as well as modeling conditional means and conditional variances within the model framework.

It is widely recognized that volatility of returns responds differently to bad news and good news. In particular, bad news tends to increase the future volatility while same-sized good news will only increase the future volatility by a smaller amount, or even cause decrease in the future volatility. The news impact function (NIF) is a powerful tool for analyzing the volatility asymmetry for GARCH-type models. The idea of the NIF is to examine the relationship between conditional volatility in period $t+1$ and the standardized shock to returns in period t in isolation. The asymmetric effect in volatility is that the effects of positive returns on volatility are different from those of negative returns of a similar magnitude and leverage refers to the negative correlation between

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the current return and future volatility. Therefore leverage denotes asymmetry, but not all asymmetric effects display leverage.

2.2 Wishart process

According to Benabid, Bensusan, El Karoui, et al. (2010), the Wishart process is an extension of a multidimensional Cox- Ingersoll-Ross process. Thus, the Wishart volatility model is also an extension of a multidimensional Heston model by considering diagonal matrices. There is a bijection concerning the volatility parameters between the Heston model and the Wishart volatility model with this restricted specification. The Wishart volatility model in the general case has additional properties. Indeed, in a multidimensional Heston model, the correlation is stochastic but depends on factors that generates the volatility dynamics. In the case of the Wishart volatility model, the correlation depends on the volatility factors but depends also on the non diagonal components of the Wishart process. It is an important property allowing degrees of freedom for the correlation and consequently for the skew and the smile. For example, in the Heston model, the change of sign of the skew is constrained by the correlation coefficients and the volatility factors whereas in the Wishart volatility model, the change of sign does not have this constraint.

Wishart processes belong to the class of affine processes and they generalise the notion of positive factor insofar as they are defined on the set of positive semidefinite real $d \times d$ matrices, denoted by S_d^+, Gnoatto (2012). Bru proved many interesting properties of the Wishart process, like non-collision of the eigenvalues (when $\alpha = n + 1$) and the additivity property shared with square Bessel processes. Moreover, she (2013) computed the Laplace transform of the Wishart process and its integral (the Matrix Cameron-Martin formula using her terminology), which plays a central role in the applications. Positive (semi)definite matrices arise in finance in a natural way and the

nice analytical properties of affine processes on S_n^+ opened the door to new interesting models which are able to overcome the shortcomings of previous affine models. In fact, the non linearity of S_n^+ is the key ingredient that allows for non trivial correlations among positive factors.

The method based on analysis of the conditional characteristic function of the log-price given volatility level was introduced by Kang and Kang (2013). In particular, they found an explicit expression for the conditional characteristic function for the Heston model and also perform numerical experiments to demonstrate the performance and accuracy of their method. They derived the conditional Laplace transform of log-price Y_T given the terminal volatility X_T using the affine transform formula and the change of measure techniques. She (2013) presented two kinds of coupled Wishart process to model volatility that is the homogenous coupled Wishart process and heterogenous coupled Wishart process. The authors developed corresponding algorithms based on the models. The homogenous coupled Wishart process refers to model that our target objects belong to the same category. A two-chain coupled Wishart process was introduced. Within such a model, the matrix in one chain is not only related with the past one from its own chain but also from its neighbors. Unlike the homogenous one, in such a model, the covariance matrices are coupled with vectors, scalars or even a system. They modelled how the outside influence from other kinds of data affect the evolving of covariance matrices and made a simplified setup to illustrate how the heterogeneous coupling works and constructed the learning algorithm based on the setups and tested it on synthetic data.

Da Fonseca, Gnoatto, and Grasselli (2015), priced for different affine stochastic volatility models some derivatives that recently appeared in the market. These products were characterised by payoffs depending on both stock and its volatility. Using a Fourier-analysis approach, they recovered in a much simpler way some results already established in the literature for the single factor specification of the volatility and provided closed-form solution for different products and two multivariate Wishart-based

stochastic volatility models. They implemented the formulas for realistic model parameter values and put results in the broader perspective of model risk.

From the existing literatures, there is still a gap on Wishart Stochastic Volatility Models with applications to emerging financial markets data. This thesis, therefore, contributes and extends the existing literature on stochastic models specifically on Wishart Stochastic Volatility Models with applications to emerging financial markets data.

Chapter 3

Emerging Market Derivatives Pricing

The World Bank classifies countries on the basis of their Gross National Income per capita into low, middle and high income economies, Lehkonen (2014). According to Lehkonen(2014), emerging market economies can be classified as countries with low to middle per capita income. The term refers to a situation in which economy emerges from a lower income per capita level to higher level that is from developing to developed economy. It should be noted that the definition of the emerging market does not necessary mean that the country is small or poor. For example such economic giants as China and Russia are classified as emerging. Often the term emerging markets refers, but is not limited, to the countries which opened their financial markets during late 1980s and early 1990s providing access for foreign investors to domestic markets as well as allowing domestic investors to trade in international markets. According to De Santis et al. (1997), an emerging financial market is a term that investors use to describe a country in which investments would be expected to a achieve higher returns but accompanied by greater risks.

The following are the characteristics of an emerging financial market:

1. Expected returns are independent of any market risk measures which means that it is not possible to dependent on the market risk measures to estimate the returns

which is very possible in the developed markets.

2. Interest rate is volatile which means that interest rates are not stable.They keep changing every time.

3. Returns are higher. Returns are the benefit to an investor. High returns means the investment gains compare favorably to investment cost. As a performance measure, returns are used to evaluate the efficiency of an investment or to compare the efficiency of a number of different investments. In purely economic terms, it is one way of considering profits in relation to capital invested.

3.1   Derivative pricing under one risky asset

Consider a financial market consisting of a risk-free asset S_0 and a risky asset S. The price dynamic of the standard risk free asset with price process S_0 is given by;

$$dS_0(t) = r(t)S_0(t)dt \tag{3.1}$$

where the short rate $r(t)$ is allowed to be either a deterministic process or a stochastic process.

The price dynamic of the standard risky asset with price process S is given by;

$$dS(t) = \mu S(t)dt + S(t)\sigma dW(t) \tag{3.2}$$

where $W(t)$ is a wiener process. Applying Ito's lemma to the function $C = C[S, t]$, gives the option dynamic as:

$$dC = \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}dS + \frac{\sigma^2}{2}S^2\frac{\partial^2 C}{\partial S^2}dt \tag{3.3}$$

Consider a portfolio given by;

$$\Pi = -C + bS$$

The dynamics of this portfolio are;

$$d\Pi = -dC + bdS \qquad (3.4)$$

Substituting equation (3.3) into equation (3.4) gives ;

$$d\Pi = \left(-\frac{\partial C}{\partial t} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 C}{\partial S^2} \right) dt + \left(b - \frac{\partial C}{\partial S} \right) dS \qquad (3.5)$$

Substituting equation (3.2) into equation (3.5);

$$d\Pi = \left(-\frac{\partial C}{\partial t} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 C}{\partial S^2} + \left[b - \frac{\partial C}{\partial S} \right] \mu S \right) dt + \left(b - \frac{\partial C}{\partial S} \right) \sigma S dW \qquad (3.6)$$

Let $b = \frac{\partial C}{\partial S}$, So;

$$d\Pi = \left(-\frac{\partial C}{\partial t} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 C}{\partial S^2} \right) dt$$

For the absence of arbitrage, it must hold that;

$$d\Pi = r\Pi dt$$

So

$$\frac{\partial C}{\partial t} + rS \frac{\partial C}{\partial S} + \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 C}{\partial S^2} - rC = 0 \qquad (3.7)$$

Equation (3.7) is a second-order partial differential equation (PDE). What determines how the equation applies to a particular derivative is given by the boundary condition.

For the call option, the boundary condition at maturity is given by;

$$C(S(T), T) = \text{Max}(S(T) - K, 0) \tag{3.8}$$

where K is the exercise price.

According to Björk (2009), solving the second order differential equation (3.7) together with the boundary condition in equation (3.8) gives the Black-Scholes formula. Let $S(t) = s$ where t is the initial time;

$$C(t, s) = sN(d_1) - e^{-r(T-t)}KN(d_2)$$

$$d_1(t, s) = \frac{1}{\sigma\sqrt{T-t}}\left[\ln(\frac{s}{K}) + [r - \frac{1}{2}\sigma^2](T-t)\right]$$

$$d_2(t, s) = d_1(t, s) - \sigma\sqrt{T-t}$$

3.2   Multidimensional derivative pricing

In this section, a generalization of the Black Scholes model to the case where apart from the risk free asset, two underlying risky assets are considered. Analysis is carried out using the classical approach and pricing when the interest rates are stochastic.

Assume a financial market consisting of two risky traded assets: $S_1(t)$, $S_2(t)$. The entire asset price vector given by $S(t)$ and in matrix notation given by;

$$S(t) = \left(\begin{array}{cc} S_1(t) & S_2(t) \end{array} \right)'$$

Under the objective probability measure P, the S- dynamics are given by;

$$dS_i(t) = \mu_i S_i(t)dt + S_i(t)\sum_{j=1}^{2}\sigma_{ij}dW_j^p(t) \tag{3.9}$$

for $i = 1, 2$. W_1^p, W_2^p are independent wiener processes.

The coefficients μ_i called the vector drift term and σ_{ij} called the volatility matrix are assumed to be adapted and volatility matrix $\sigma = \{\sigma_{ij}\}_{i,j=1}^2$ is non singular (invertible). The dynamic of standard risk free asset with price process S_0, is given in equation (3.1)

Let $W^P(t)$ denote the column vector

$$W^P(t) = \left(\begin{array}{cc} W_1^P(t) & W_2^P(t) \end{array} \right)'$$

and define row vector $\sigma_i = [\sigma_{i1}, \sigma_{i2}]$

According to Contreras, Llanquihuén, and Villena (2015), the price dynamics can be written more compactly as;

$$dS_i(t) = \mu_i S_i(t) dt + S_i(t)\sigma_i dW_j^P(t) \tag{3.10}$$

for $i = 1, 2$ and $j = 1, 2$

Or Come up with the two equations

$$\frac{dS_1(t)}{S_1(t)} = \mu_1 dt + \sigma_{11} dW_1^P(t) + \sigma_{12} dW_2^P(t) \tag{3.11}$$

$$\frac{dS_2(t)}{S_2(t)} = \mu_2 dt + \sigma_{21} dW_1^P(t) + \sigma_{22} dW_2^P(t) \tag{3.12}$$

Before getting the prices, equations (3.11) and (3.12) need to be transformed to risk neutral which can be done with the help of the multidimensioal Girsanov's theorem.

Theorem 3.2.1. (Papaioannou (2012, Page 4)) Multidimensional Girsanovs Theorem

Consider a filtration $F(t)$ over a period $[0, T]$ where $T < \infty$.

Let $\Lambda(t) = (\lambda_1(t), \lambda_2(t), ..., \lambda_n(t))$ be an n-dimensional process that is $F(t)$-measurable

and satisfies a condition:

$$E\left\{\exp\left[\frac{1}{2}\int_0^t \sum_{i=1}^n \lambda_i^2(s)ds\right]\right\} < \infty,$$

$t \in [0, T]$

Define a random process $M(t)$:

$$M(t) = \exp\left[\sum_{i=1}^n \left(-\int_0^t \lambda_i(s)dW_i^P(s) - \frac{1}{2}\int_0^t \lambda_i^2(s)ds\right)\right]$$

$t \in [0, T]$, where $W_i^P(t)$ for $i = 1, ..., n$ is an n-dimensional Brownian motion under probability measure P. Then under the measure Q, $W_i^Q(t)$ is a multidimensional Wiener process defined by:

$$W_i^Q(t) = W_i^P(t) + \int_0^t \lambda_i(u)du$$

$t \in 0, T$ and $i = 1, 2, ..., n$

Using Theorem 3.2.1 to transform the two equations (3.11) and (3.12) from physical measure P to risk neutral measure Q gives;

$$\frac{dS_1(t)}{S_1(t)} = rdt + \sigma_{11}dW_1^Q(t) + \sigma_{12}dW_2^Q(t) \tag{3.13}$$

$$\frac{dS_2(t)}{S_2(t)} = rdt + \sigma_{21}dW_1^Q(t) + \sigma_{22}dW_2^Q(t) \tag{3.14}$$

3.2.1   Infinitesimal generator

Given a markov process whose dynamics in vector form satisfies

$$dS(t) = K(t, S(t))dt + H(t, S(t))dW_t, S(t) \in \mathbb{R}^n \tag{3.15}$$

$S(0) = m \in \mathbb{R}^n$

$\mathbb{A}$ is called an infinitesimal generator of S if given $h \in C^{1,2}(\mathbb{R}^n)$

$$\mathbb{A}h(t, S) = \sum_{i=1}^{n} K_i(t, S(t)) \frac{\partial h(t, S(t))}{\partial S_i}(t, S) + \frac{1}{2} \sum_{j=1}^{n} (H.H')_{ij} \frac{\partial^2 h}{\partial S_i \partial S_j} \tag{3.16}$$

where $K(t, S)$ is a vector and $H(t, S)$ is a matrix.

So the infinitesimal generator $\mathbb{A}$ is given by

$$\mathbb{A} = \sum_{i=1}^{n} K_i(t, S(t)) \frac{\partial}{\partial S_i} + \frac{1}{2} \sum_{j=1}^{n} (H.H')_{ij} \frac{\partial^2}{\partial S_i \partial S_j}$$

If $h \in C^2(\mathbb{R})$ then equation (3.16) become:

$$\mathbb{A}h(t, S) = K(t, S(t)) \frac{\partial}{\partial S} + \frac{1}{2} (H^2) \frac{\partial^2}{\partial S^2}$$

Theorem 3.2.2. (Björk (2009, Page 75)) Multi-Dimensional Feynman-kac formula

Assume F is a solution to the boundary value problem

$$\frac{\partial F}{\partial t}(t, S) + \sum_{i=1}^{n} K_i(t, S(t)) \frac{\partial F(t, S(t))}{\partial S_i}(t, S) + \frac{1}{2} \sum_{j=1}^{n} (H.H')_{ij} \frac{\partial^2 F}{\partial S_i \partial S_j} - rF = 0$$

$$\tag{3.17}$$

with the boundary condition

$F(T, S) = \phi(S) = \phi(S_1, ..., S_n)$ and where S satisfies the stochastic differential equation (3.15)

Then F has a representation

$$F(t, S) = e^{-r(T-t)} E_{t,s}(\phi(S(T))) \tag{3.18}$$

From Theorem(3.2.2), the Feynman-kac formula of the two equations (3.13) and (3.14) becomes:

$$\frac{\partial F}{\partial t} + rS_1 \frac{\partial F}{\partial S_1} + rS_2 \frac{\partial F}{\partial S_2} + \frac{1}{2} \left(S_1^2 C_{11} \frac{\partial^2 F}{\partial S_1^2} + 2S_1 S_2 C_{12} \frac{\partial^2 F}{\partial S_1 \partial S_2} + S_2^2 C_{22} \frac{\partial^2 F}{\partial S_2^2} \right) - rF = 0$$
$$\tag{3.19}$$

where

$$HH' = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \begin{pmatrix} \sigma_{11} & \sigma_{21} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} \sigma_{11}^2 + \sigma_{12}^2 & \sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22} \\ \sigma_{21}\sigma_{11} + \sigma_{22}\sigma_{12} & \sigma_{21}^2 + \sigma_{22}^2 \end{pmatrix}$$

$$= \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

$$= \frac{\partial F}{\partial t} + r \left(S_1 \frac{\partial F}{\partial S_1} + S_2 \frac{\partial F}{\partial S_2} \right) + \frac{1}{2} \left(S_1^2 C_{11} \frac{\partial^2 F}{\partial S_1^2} + 2S_1 S_2 C_{12} \frac{\partial^2 F}{\partial S_1 \partial S_2} + S_2^2 C_{22} \frac{\partial^2 F}{\partial S_2^2} \right) - rF = 0$$

with the boundary condition $F(T, S) = \phi(S) = \phi(S_1, S_2)$

The exchange option payoff $\phi[S_1(T), S_2(T)] = [S_1(T) - S_2(T)]^+$

Remark

Black scholes formula can not be used because the strike price is stochastic.

Assume that the contract function ϕ is homogeneous of degree 1, and that the volatility

matrix σ is constant. Then the pricing function F is given by:,

$$F(T, S) = \phi(S) = (S_1 - S_2)^+ = S_2 \left(\frac{S_1}{S_2} - 1 \right)^+$$

$$F_t = S_2 G_t$$

$$F_{S_1} = S_2 G_{S_1}(T, Z) = S_2 G_Z Z_{S_1}$$

$$F_{S_2} = G + S_2.G_Z.\left(-\frac{S_1}{S_2^2} \right)$$

$$F_{S_1 S_1} = G_{ZZ}.\frac{1}{S_2}$$

$$F_{S_1 S_2} = G_{ZZ}.\left(-\frac{S_1}{S_2^2} \right)$$

$$F_{S_2 S_2} = G_{ZZ}.\frac{S_1^2}{S_2^3}$$

Substituting in equation (3.19) gives:

$$G_t + 0 + \frac{1}{2}G_{ZZ}Z^2(C_{11} + C_{22} - 2C_{12})$$

$G(T, S) = (Z - 1)^+$, where $\sigma_z = \sqrt{C_{11} + C_{22} - 2C_{12}}$

G is given by Black Scholes formula:

$$G(t, Z) = ZN(d_1^Z) - e^0[1.N(d_2^Z)]$$

where the strike price is 1, T is the time of maturity and N is the cumulative density function for a normal distribution $N(0, 1)$

$$d_1^Z = \frac{1}{\sigma_Z\sqrt{T-t}} \left(\ln\frac{Z}{1} + \frac{1}{2}(\sigma^Z)^2(T-t) \right)$$

$$d_2^Z = d_1^Z - \sigma^Z(T-t)$$

The price of the exchange option

$$F(t, S_1, S_2) = S_2 G\left(t, \frac{S_1}{S_2}\right) = S_2\left(\frac{S_1}{S_2}N(d_1^Z) - N(d_2^Z)\right) = S_1 N(d_1^Z) - S_2 N(d_2^Z)$$

3.2.2 Stochastic Interest rates

In this subsection interest rates are considered to be stochastic. Stochastic Interest rates follow a certain random process and are denoted by $r(t)$.

Definition 3.2.1. A zero coupon bond with maturity date T, also called a T-bond, is a contract which guarantees the holder 1 dollar to be paid on the date T. The price at time t of a bond with maturity date T is denoted by $p(t, T)$ and given by;

$$P(t, T) = E_t^Q(e^{-\int_t^T r(s)ds})$$

The dynamics of $r(t)$ can be given by:

1. Vasicek interest rate model : $dr(t) = [a - br(t)]dt + \sigma_r dW^Q(t)$ which has low probability that interest rate will be negative and applies to short term normally distributed interest rates.

2. Cox Ingersoll Ross (CIR) model: $dr(t) = [a - br(t)]dt + \sigma_r[\sqrt{r(t)}]dW^Q(t)$ which has positive interest rates and applies to long time non-gaussian distributed interest rates.

Both models that is Vasicek interest rate model and Cox Ingersoll Ross model have affine term structure which in this case means that the bond price can be written in such a way that the logarithm of the bond price is an affine function of r.

3.2.3 Affine term structure

Definition 3.2.2. According to Björk (2009), if the term structure $p(t, T); 0 \le t \le T, T > 0$ has the form

$$p(t, T) = F(t, r(t), T)$$

where F has the form

$$F(t, r(t), T) = e^{A(t,T)-B(t,T)r}$$

where A and B are deterministic functions then the model is said to possess an affine term structure (ATS).

Proposition 3.2.1. Consider the $Q-$ dynamics

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dw(t)$$

Assume that μ and σ are of the form

$$\mu(t, r) = \alpha(t)r + \beta(t)$$

$$\sigma(t, r) = \sqrt{\gamma(t)r + \delta(t)}$$

Then the model admits an ATS of the form $F(t, r(t), T) = e^{A(t,T)-B(t,T)r}$, where A and B satisfy the system

$$B_t(t, T) + \alpha(t)B(t, T) - \frac{1}{2}\gamma(t)B^2(t, T) = -1$$

$$A_t(t, T) = \beta(t)B(t, T) - \frac{1}{2}\delta(t)B^2(t, T)$$

$$B(T, T) = 0, A(T, T) = 0$$

Therefore from the proposition (3.2.1), the ATS of the dynamics:

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW(t) \tag{3.20}$$

exists if μ and σ^2 are both affine that is linear plus a constant functions of r.

Vasicek interest rate model: $dr(t) = (a - br(t))dt + \sigma_r dW^Q(t)$ **has an ATS**

proof 3.2.1. The term structure equation of the model is given by:

$$\frac{\partial F}{\partial t} + (a - br(t))\frac{\partial F}{\partial r} + \frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial r^2} - rF = 0 \tag{3.21}$$

$F(T, r) = 1$

Since μ and σ^2 are affine then $F(t, r(t), T) = e^{A(t,T)-B(t,T)r}$

Therefore:

$$\frac{\partial F}{\partial t} = F_t = (-B_t(t, T)r + A_t(t, T))F$$

$$\frac{\partial F}{\partial r} = F_r = -B(t, T)F$$

$$\frac{\partial^2 F}{\partial r^2} = F_{rr} = B^2(t, T)F$$

Substituting in equation (3.21), gives;

$$B_t(t, T) - aB(t, T) = -1 \tag{3.22}$$

$$A_t(t, T) = bB(t, T) - \frac{1}{2}\sigma^2 B^2(t, T) \tag{3.23}$$

$B(T, T) = 0, A(T, T) = 0$

Equation (3.22) is, for each fixed T, a simple linear ODE in the t-variable which can be solved.

$$B_t(t, T) - aB(t, T) = -1$$

Let $B(t, T)$ be Y

Then equation (3.22) become:

$$\frac{dY}{dt} + aY = -1 \tag{3.24}$$

The integrating factor is given by $e^{\int a dt} = e^{at}$

Multiplying equation (3.24) by the integrating factor, gives;

$$\frac{dY}{dt} \cdot e^{at} + aY \cdot e^{at} = -1 \cdot e^{at} \tag{3.25}$$

The left hand side of the equation (3.25) is the same as $\frac{d}{dt}(Ye^{at})$

Therefore:

$$\frac{d}{dt}(Ye^{at}) = -e^{at}$$

$$Ye^{at} = \int_t^T -e^{au} du$$

$$Ye^{at} = -\frac{1}{a}[e^{aT} - e^{at}]$$

$$Y = -\frac{1}{a}\frac{e^{aT} - e^{at}}{e^{at}}$$

$$Y = -\frac{1}{a}(e^{a(T-t)} - 1)$$

Which implies that:

$$B(t,T) = -\frac{1}{a}(e^{a(T-t)} - 1)$$

$B(t,T)$ is substituted in equation (3.23) to get $A(t,T)$.

so $A(t,T)$ become:

$$A(t,T) = b\int_t^T B(u,T)du - \frac{1}{2}\sigma^2\int_t^T B^2(u,T)du$$

3.2.4 Zero Coupon bond pricing using Cox-Ingersoll-Ross(CIR) model

Cox-Ingersoll-Ross(CIR) model also called squared bessel process is given by:

$$dr(t) = (a - br(t))dt + \sigma(\sqrt{r(t)})dW^Q(t)$$

Suppose that

$$P(t,T) = E_t^Q(e^{-\int_t^T r(s)ds}) = F(t,r,T)$$

The term structure equation of the model is given by:

$$\frac{\partial F}{\partial t} + (a - br(t))\frac{\partial F}{\partial r} + \frac{1}{2}\sigma^2 r\frac{\partial^2 F}{\partial r^2} - rF = 0 \qquad (3.26)$$

$F(T,r) = 1$

Since the model is affine then:

$$P(t,T) = E_t^Q(e^{-\int_t^T r(s)ds}) = F(t,r,T) = e^{A(t,T)-B(t,T)r} \qquad (3.27)$$

From equation (3.27):

$$\frac{\partial F}{\partial r} = -BF$$

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$$\frac{\partial^2 F}{\partial r^2} = B^2 F$$

$$\frac{\partial F}{\partial t} = \left(\frac{\partial A}{\partial t} - \frac{\partial B}{\partial t}r\right)F$$

Substituting in equation (3.26),gives:

$$\left(\frac{\partial A}{\partial t} - \frac{\partial B}{\partial t}r\right)F - BF(a - br) + \frac{1}{2}B^2 F\sigma_2^2 r - rF = 0 \tag{3.28}$$

Therefore the Riccati ordinary differential equations of the equation (3.28) are given

by

$$-\frac{\partial B}{\partial t} + bB + \frac{1}{2}\sigma_2^2 B^2 - 1 = 0 \tag{3.29}$$

$$\frac{\partial A}{\partial t} - aB = 0 \tag{3.30}$$

$$A(T,T) = B(T,T) = 0$$

Solving equations (3.29) and (3.30), gives A and B which is substituted in equation

(3.27) to get the price.

The CIR model can be seen as the square of Vasicek interest rate model. Thus consid-

ering the equality

CIR model = Vasicek interest rate model where the Vasicek interest rate model is given

by

$$dS(t) = (a - bS)dt + \sigma dW(t)$$

Let $\Sigma = S^2$

Then let $f = \Sigma = S^2$, it implies that $f_t = 0$, $f_s = 2s$, $f_{ss} = 2$

Applying Ito's formula gives;

$$df = d\Sigma = 0dt + 2sds + \frac{1}{2}2dt$$

$$d\Sigma = 2sds + 1dt$$

But $S = \sqrt{\Sigma}$ and $S = W$. So:

$$d\Sigma = 2\sqrt{\Sigma}dW + 1dt$$

Therefore for CIR model to be the square of Vasicek interest rate model,

The parameters of CIR must be as below:

$a = 1$, $b = 0$ and $\sigma = 2$

If $S \in \mathbb{R}$ then $\Sigma \in \mathbb{R} > 0$ since $\Sigma = S^2$. And if $S \in \mathbb{R}^2$, is $\Sigma \in \mathbb{R}^2 > 0$?

But if $S \geq 0$ then there exist $S \in \mathbb{R}^2$, such that $S^2 = \Sigma$

$\Rightarrow S.S = \Sigma$, which is not a well defined product.

So a vector product such that the result is another vector is defined

Consider $S \in Sym_{2 \times 2}$ that is $S = S'$. Recall that if $S = W$ and $\Sigma = S^2$ then

$d\Sigma = 2\sqrt{\Sigma}dW + 1dt$ (square root process), this idea is extended.

Let $S = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}$ where $W_{ij} \neq W_{em} \; \forall (i,j) \neq (e,m)$

Let $S(t).S'(t) = \Sigma(t)$ with $dS(t) = MSdt + QdW(t)$ an Ornstein-Uhlenbeck process.
Then:

$$d\Sigma(t) = \Sigma(t + dt) - \Sigma(t)$$

$$d\Sigma(t) = S(t + dt).S'(t + dt) - S(t).S'(t)$$

$$d\Sigma(t) = (S(t) + MSdt + QdW(t))(S(t) + MSdt + QdW(t))' - S(t).S'(t)$$

But $dt.dt = 0$, $dW.dt = 0$ and $dW.dW = dt$
Therefore:

$$d\Sigma(t) = S(t).S'(t)M'dt + MS(t).S'(t)dt + KQQ'dt + SdW'Q' + QdWS'$$

$$dΣ = (KQ'Q + MΣ(t) + Σ(t)M')dt + Q\sqrt{Σ}(t)dW + dW'\sqrt{Σ}(t)Q' \qquad (3.31)$$

Equation (3.31) is called Wishart stochastic differential equation where $K \in \mathbb{R} > 0$, $M \in M_{n \times n}$ and $Q \in M_{n \times n}$.

Equation (3.31) can be written as:

$$dΣ = (ΩΩ' + MΣ(t) + Σ(t)M')dt + Q\sqrt{Σ}(t)dW + dW'\sqrt{Σ}(t)Q' \qquad (3.32)$$

where

$ΩΩ' = KQ'Q$, $Ω$ is invertible

Computations in the two dimensional case for the Wishart stochastic differential equation(3.32) are developed. This means that the Wishart process $Σ_t$ satisfies the following SDE:

$$dΣ = d\begin{pmatrix} Σ_{11} & Σ_{12} \\ Σ_{21} & Σ_{22} \end{pmatrix} = \left[\begin{pmatrix} Ω_{11} & Ω_{12} \\ Ω_{21} & Ω_{22} \end{pmatrix} \begin{pmatrix} Ω_{11} & Ω_{21} \\ Ω_{12} & Ω_{22} \end{pmatrix} + \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} Σ_{11} & Σ_{12} \\ Σ_{21} & Σ_{22} \end{pmatrix} \right] dt$$

$$+ \left[\begin{pmatrix} Σ_{11} & Σ_{12} \\ Σ_{21} & Σ_{22} \end{pmatrix} \begin{pmatrix} M_{11} & M_{21} \\ M_{12} & M_{22} \end{pmatrix} \right] dt + \sqrt{\begin{pmatrix} Σ_{11} & Σ_{12} \\ Σ_{21} & Σ_{22} \end{pmatrix}} \begin{pmatrix} dW_{11} & dW_{12} \\ dW_{21} & dW_{22} \end{pmatrix} \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$$

$$+ \begin{pmatrix} q_{11} & q_{21} \\ q_{12} & q_{22} \end{pmatrix} \begin{pmatrix} dW_{11} & dW_{21} \\ dW_{12} & dW_{22} \end{pmatrix} \sqrt{\begin{pmatrix} Σ_{11} & Σ_{12} \\ Σ_{21} & Σ_{22} \end{pmatrix}} \qquad (3.33)$$

Let:

$$
\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} = \sqrt{\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}} = \sqrt{\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12} & \Sigma_{22} \end{pmatrix}} \quad (3.34)
$$

This is because Σ is symmetric.

So from equation (3.34);

$$
\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12} & \Sigma_{22} \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}^2 = \begin{pmatrix} \sigma_{11}^2 + \sigma_{12}^2 & \sigma_{11}\sigma_{12} + \sigma_{12}\sigma_{22} \\ \sigma_{11}\sigma_{12} + \sigma_{12}\sigma_{22} & \sigma_{12}^2 + \sigma_{22}^2 \end{pmatrix}
$$

$$
d\Sigma = d\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12} & \Sigma_{22} \end{pmatrix} = \left[\begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix} \begin{pmatrix} \Omega_{11} & \Omega_{21} \\ \Omega_{12} & \Omega_{22} \end{pmatrix} + \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12} & \Sigma_{22} \end{pmatrix} \right] dt
$$

$$
+ \left[\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12} & \Sigma_{22} \end{pmatrix} \begin{pmatrix} M_{11} & M_{21} \\ M_{12} & M_{22} \end{pmatrix} \right] dt + \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \begin{pmatrix} dW_{11} & dW_{12} \\ dW_{21} & dW_{22} \end{pmatrix} \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}
$$

$$
+ \begin{pmatrix} q_{11} & q_{21} \\ q_{12} & q_{22} \end{pmatrix} \begin{pmatrix} dW_{11} & dW_{21} \\ dW_{12} & dW_{22} \end{pmatrix} \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \quad (3.35)
$$

$$
d\Sigma_{11} = (\Omega_{11}^2 + \Omega_{12}^2 + 2M_{11}\Sigma_{11})dt + 2\sigma_{11}(q_{11}dW_{11} + q_{21}dW_{12}) + 2\sigma_{12}(q_{11}dW_{12} + q_{21}dW_{22}) \quad (3.36)
$$

Since $(dt)^2 = 0$, $(dW)^2 = dt$ then:

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$$d < \Sigma_{11}, \Sigma_{11} >= (d\Sigma_{11})^2 = 4\Sigma_{11}(q_{11}^2 + q_{21}^2)dt$$

$$d\Sigma_{11} = (\Omega_{11}^2 + \Omega_{12}^2 + 2M_{11}\Sigma_{11})dt + \alpha\sqrt{\Sigma_{11}}d\tilde{W}$$

$$d < \Sigma_{22}, \Sigma_{22} >= 4\Sigma_{22}(q_{12}^2 + q_{22}^2)dt$$

$$d < \Sigma_{11}, \Sigma_{22} >= 4\Sigma_{12}(q_{11}q_{12} + q_{21}q_{22})dt$$

These will be applied in the generating of the infinitesimal generator of the process.

3.2.5 Infinitesimal generator of Wishart process

The infinitesimal generator is used in the getting of the moment generating function. Recall that if

$$dS(t) = K(t, S(t))dt + H(t, S(t))dW_t$$

then;

$$\mathbb{A} = K\frac{\partial}{\partial S} + \frac{1}{2}H^2\frac{\partial^2}{\partial S^2}$$

Consider;

$$d\Sigma = (\Omega\Omega' + M\Sigma(t) + \Sigma(t)M')dt + Q\sqrt{\Sigma}(t)dW + dW'\sqrt{\Sigma}(t)Q' \qquad (3.37)$$

The infinitesimal generator of equation (3.37) is given by;

$$\mathbb{A} = Tr[(\Omega\Omega' + M\Sigma(t) + \Sigma(t)M')D + 2\Sigma DQ'QD] \qquad (3.38)$$

where;

$$Tr(2\Sigma DQ'QD) = 2Tr(\Sigma DQ'QD)$$

and;

$$D = \begin{pmatrix} \frac{\partial}{\partial \Sigma_{11}} & \frac{\partial}{\partial \Sigma_{12}} \\ \frac{\partial}{\partial \Sigma_{21}} & \frac{\partial}{\partial \Sigma_{22}} \end{pmatrix} \tag{3.39}$$

D is a matrix differential operator and not symmetric

$$2\Sigma DQ'QD = 2 \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12} & \Sigma_{22} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial \Sigma_{11}} & \frac{\partial}{\partial \Sigma_{12}} \\ \frac{\partial}{\partial \Sigma_{21}} & \frac{\partial}{\partial \Sigma_{22}} \end{pmatrix} \begin{pmatrix} q_{11} & q_{21} \\ q_{12} & q_{22} \end{pmatrix} \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial \Sigma_{11}} & \frac{\partial}{\partial \Sigma_{12}} \\ \frac{\partial}{\partial \Sigma_{21}} & \frac{\partial}{\partial \Sigma_{22}} \end{pmatrix} \tag{3.40}$$

After some computations gives:

$$Tr(2\Sigma DQ'QD) = \frac{1}{2}\{< \Sigma_{11}, \Sigma_{11} > \frac{\partial^2}{\partial \Sigma_{11}^2} + 4 < \Sigma_{12}, \Sigma_{12} > \frac{\partial^2}{\partial \Sigma_{12}^2} + < \Sigma_{22}, \Sigma_{22} > \frac{\partial^2}{\partial \Sigma_{22}^2}$$

$$+4 < \Sigma_{11}, \Sigma_{12} > \frac{\partial^2}{\partial \Sigma_{11}\partial \Sigma_{12}} + 2 < \Sigma_{11}, \Sigma_{22} > \frac{\partial^2}{\partial \Sigma_{11}\partial \Sigma_{22}} + 4 < \Sigma_{12}, \Sigma_{22} > \frac{\partial^2}{\partial \Sigma_{12}\partial \Sigma_{22}}\} \tag{3.41}$$

3.2.6 Derivative pricing using a Wishart process

Computing the moment generating function given in equation (3.42) below is needed.

$$\varphi(t, \Lambda, \tilde{\Lambda}) = E_t(e^{Tr(\Lambda \Sigma_T) + Tr \int_t^T \tilde{\Lambda}\Sigma_s ds}) \tag{3.42}$$

But Wishart process being affine gives;

$$\varphi(t, \Lambda, \tilde{\Lambda}) = F(t, \Sigma) = e^{Tr(A_1(t,T)\Sigma_t + A_2(t,T))} \tag{3.43}$$

Apply the Feynman Kac formula;

$$\frac{\partial F}{\partial t} + \mathbb{A}F + Tr(\tilde{\Lambda}\Sigma)F = 0 \tag{3.44}$$

$$F(T, \Sigma) = e^{Tr(\Lambda\Sigma)}$$

$$Tr(\frac{\partial A_1}{\partial t}\Sigma + \frac{\partial A_2}{\partial t})F + Tr[(\Omega\Omega' + M\Sigma + \Sigma M')D + 2\Sigma DQ'QD]F + Tr(\tilde{\Lambda}\Sigma)F = 0 \tag{3.45}$$

$$Tr(\frac{\partial A_1}{\partial t}\Sigma + \frac{\partial A_2}{\partial t})F + Tr[(\Omega\Omega' + M\Sigma + \Sigma M')DF + (2\Sigma DQ'QD)F] + Tr(\tilde{\Lambda}\Sigma)F = 0 \tag{3.46}$$

That is

$$De^{Tr(A_1\Sigma_t + A_2)} = \begin{pmatrix} \frac{\partial(e^{Tr(A_{11}\Sigma_{11}+A_2)})}{\partial\Sigma_{11}} & \frac{\partial(e^{Tr(A_{12}\Sigma_{12}+A_2)})}{\partial\Sigma_{12}} \\ \frac{\partial(e^{Tr(A_{21}\Sigma_{21}+A_2)})}{\partial\Sigma_{21}} & \frac{\partial(e^{Tr(A_{22}\Sigma_{22}+A_2)})}{\partial\Sigma_{22}} \end{pmatrix}$$

$$= \begin{pmatrix} A_{11}e^{Tr(A_{11}\Sigma_{11}+A_2)} & A_{12}e^{Tr(A_{12}\Sigma_{12}+A_2)} \\ A_{21}e^{Tr(A_{21}\Sigma_{21}+A_2)} & A_{22}e^{Tr(A_{22}\Sigma_{22}+A_2)} \end{pmatrix}$$

$$Tr(\frac{\partial A_1}{\partial t}\Sigma + \frac{\partial A_2}{\partial t}) + Tr[(\Omega\Omega' + M\Sigma + \Sigma M')A_1 + 2\Sigma A_1 Q'QA_1] + Tr(\tilde{\Lambda}\Sigma) = 0 \tag{3.47}$$

$$Tr(\Sigma(\frac{\partial A_1}{\partial t} + A_1 M + M'A_1 + 2A_1 Q'QA_1 + \tilde{\Lambda})) + Tr(\frac{\partial A_2}{\partial t} + \Omega\Omega' A_1) = 0 \tag{3.48}$$

The matrix Riccati equations are:

$$\frac{\partial A_1}{\partial t} + A_1 M + M'A_1 + 2A_1 Q'QA_1 + \tilde{\Lambda} = 0 \tag{3.49}$$

$$A_1(T, T) = \Lambda$$

$$\frac{\partial A_2}{\partial t} + \Omega\Omega' A_1 = 0 \tag{3.50}$$

$$A_2(T, T) = 0$$

Implying

$$\int_t^T A_2(T) = -\int_t^T \Omega\Omega' A_1 ds$$

$$A_2(t,T) = \int_t^T \Omega\Omega' A_1 ds \qquad (3.51)$$

Solve for A_1 by Riccati matrix linearlization

Let $A_1 = H^{-1}G$

Then $HA_1 = G$

$$\frac{\partial G}{\partial t} = \frac{\partial H}{\partial t}A_1 + \frac{\partial A_1}{\partial t}H \qquad (3.52)$$

$$\frac{\partial G}{\partial t} = \frac{\partial H}{\partial t}A_1 - H(A_1 M + M'A_1 + 2A_1 Q'QA_1 + \tilde{\Lambda}) \qquad (3.53)$$

$$\frac{\partial G}{\partial t} = \frac{\partial H}{\partial t}A_1 - HA_1 M - HM'A_1 - 2HA_1 Q'QA_1 - H\tilde{\Lambda} \qquad (3.54)$$

$$\frac{\partial G}{\partial t} = \frac{\partial H}{\partial t}A_1 - GM - HM'A_1 - 2GQ'QA_1 - H\tilde{\Lambda} \qquad (3.55)$$

Implying

$$\frac{\partial H}{\partial t}A_1 - \frac{\partial G}{\partial t} = \begin{pmatrix} \frac{\partial H}{\partial t} & \frac{\partial G}{\partial t} \end{pmatrix} \begin{pmatrix} A_1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} H & G \end{pmatrix} \begin{pmatrix} M' & -\tilde{\Lambda} \\ 2Q'Q & -M \end{pmatrix} \begin{pmatrix} A_1 \\ -1 \end{pmatrix} \begin{pmatrix} \frac{\partial H'}{\partial t} \\ \frac{\partial G'}{\partial t} \end{pmatrix}$$

$$= \begin{pmatrix} M & 2QQ' \\ -\tilde{\Lambda}' & -M' \end{pmatrix} \begin{pmatrix} H' \\ G' \end{pmatrix}$$

$$H(T) = 1, G(T) = \Lambda$$

Recall that given:

$\dot{X} = aX$, with $X(T) = b$

$$X_T = e^{(T-t)a}X_t$$
$$X_t = e^{-(T-t)a}X_T$$

So in matrix case,

$$\begin{pmatrix} H' \\ G' \end{pmatrix}(t) = e^{-(T-t)\begin{pmatrix} M & 2QQ' \\ -\tilde{\Lambda}' & -M' \end{pmatrix}} \begin{pmatrix} H'(T) \\ G'(T) \end{pmatrix} \tag{3.56}$$

$$\begin{pmatrix} H' \\ G' \end{pmatrix}(t) = e^{-(T-t)\begin{pmatrix} M & 2QQ' \\ -\tilde{\Lambda}' & -M' \end{pmatrix}} \begin{pmatrix} 1 \\ \Lambda' \end{pmatrix} \tag{3.57}$$

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So after getting the values of G and H, A_1 can be got.

3.3 Wishart Stochastic Volatility Model

According to Da Fonseca et al. (2015), Wishart Stochastic Volatility Model extends the original Heston (1993) model to the case where the volatility is described by the Wishart process, a matrix-valued stochastic process introduced by Bru (1991). Within the model the dynamics for the price are given by the following SDE:

$$dS_t = S_t \mu dt + S_t Tr(\sqrt{\Sigma_t}(dW_t R' + dB_t \sqrt{1 - RR'})) \qquad (3.58)$$

where $W_t, B_t \in M_n$ (the set of square matrices) are composed by n^2 independent Brownian motions under the risk-neutral measure (B_t and W_t are independent), $R \in M_n$ represents the correlation matrix and Σ_t belongs to the set of symmetric $n \times n$ positive semi-definite matrices. In this specification the volatility is multi-dimensional and depends on the elements of the matrix process Σ_t, which is assumed to satisfy the dynamics given in the stochastic differential equation (1.1). However, the existing Wishart Stochastic Volatility model does not cater for some of the characteristics of the emerging financial markets like interest rate being volatile. Basing on these characteristics, a modified model was got which accounts for risk premium (the volatility matrix in the drift term of the price dynamic accounts for risk premium). The joint dynamics of $\log S_t$ and Σ_t is given by the stochastic differential system:

$$d \log S_t = [\mu + (Tr(D_1 \Sigma_t), ..., Tr(D_n \Sigma_t))']dt + \sqrt{\Sigma_t} dW_t^s \qquad (3.59)$$

$$d\Sigma_t = (KQ'Q + M\Sigma_t + \Sigma_t M')dt + \sqrt{\Sigma_t} dW_t^\sigma \sqrt{(Q'Q)} + \sqrt{(Q'Q)}(dW_t^\sigma)' \sqrt{\Sigma_t} \quad (3.60)$$

where W_t^s and W_t^σ are a n-dimensional vector and a (n, n) matrix, respectively, whose elements are independent unidimensional standard Brownian motions, μ is a constant n-dimensional vector, K is a scalar such that $K > n - 1$, and $D_i, i = 1, ..., n, M, Q$

are (n, n) matrices with $Q'Q$ invertible. $\Sigma_t^{\frac{1}{2}}$ is the positive symmetric square root of the volatility matrix Σ_t.

3.4   Foreign Exchange derivative pricing

In this section, the newly constructed model is applied in the pricing of European call option. But since our model is n dimensional, consider $n = 2$.

Consider the European call C to be a function of $S(t)A^*(t, T), A(t, T), X, T$ where $S(t)$ is the spot domestic currency price of a unit of foreign exchange at time t, $A^*(t, T)$ is the foreign currency price of a pure discount bond which pays one unit of foreign exchange at time $t + T$, $A(t, T)$ is the domestic currency price of a pure discount bond which pays one unit of domestic currency at time $t + T$, X is the domestic currency exercise price of an option on foreign currency, t is the initial time and T is the expiration time.

The following assumptions are to be considered

C has the general functional form

$$C = C\left[S(t)A^*(t, T), A(t, T), X, T\right]$$

subjected to the boundary conditions

$$C\left[S(t + T), X, 0\right] = \max\left[0, S(t + T) - X\right] \tag{3.61}$$

$$C\left[0, A(t, T), X, T\right] = 0 \tag{3.62}$$

Where equation (3.61) is the terminal value of the call option, which has to be greater than zero or the strike value and equation (3.62) means that when the spot exchange value is zero, then option to be bought has a zero value.

The second assumption has to do with the dynamics of S, A^*, and A. Let dW_1, dW_2, dW_3 denote standardized Wiener processes with unit instantaneous variances and correlation matrix

$$
\begin{pmatrix}
1 & \rho_{SA^*} & \rho_{SA} \\
\rho_{SA^*} & 1 & \rho_{A^*A} \\
\rho_{SA} & \rho_{A^*A} & 1
\end{pmatrix} dt
$$

where $\rho_w = \rho_w(t,T)$ can be a known function of time (t) and the time to maturity of the bond (T). Assume S, A^*, A follow the Geometric Brownian Motions

$$
\frac{dS}{S} = \mu_S(t)dt + \sigma_S(t)dW_1 \tag{3.63}
$$

$$
\frac{dA^*}{A^*} = \mu_{A^*}(t,T)dt + \sigma_{A^*}(t,T)dW_2 \tag{3.64}
$$

$$
\frac{dA}{A} = \mu_A(t,T)dt + \sigma_A(t,T)dW_3 \tag{3.65}
$$

Basing on the assumption above, new variables dH, dW_4 can be defined and using Ito's product rule,

$$
\frac{dH}{H} = \frac{d(SA^*)}{SA^*} = (\mu_S + \mu_{A^*} + \rho_{SA^*}\sigma_S\sigma_{A^*})dt + \sigma_S dW_1 + \sigma_{A^*} dW_2
$$

with a correlation coefficient between them $\rho_{SA^*} dt = dW_1 dW_2$

$$
\frac{dH}{H} = \mu_H(t,T)dt + \sigma_H(t,T)dW_4 \tag{3.66}
$$

and write the correlation matrix of dW_4, dW_3 as

$$\begin{pmatrix} 1 & \rho_{HA} \\ \rho_{HA} & 1 \end{pmatrix} dt$$

where $\rho_{HA} = \rho_{HA}(t, T)$.

Applying Ito's lemma to the function

$$C = C\left[S(t)A^*(t, T), A(t, T), X, T\right] = C\left[H(t, T), A(t, T), X, T\right]$$

gives the option dynamic as:

$$dC = \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial H}dH + \frac{\partial C}{\partial A}dA + \frac{1}{2}\left(\frac{\partial^2 C}{\partial H^2}H^2\sigma_H^2 + 2\frac{\partial^2 C}{\partial H\partial A}HA\rho_{HA}\sigma_H\sigma_A + \frac{\partial^2 C}{\partial A^2}A^2\sigma_A^2\right)dt$$

(3.67)

Let θ represent elements involving second derivative and $\tau = T - t$ then $dt = -d\tau$, so equation (3.67) becomes:

$$dC = \frac{\partial C}{\partial \tau}d\tau + \frac{\partial C}{\partial H}dH + \frac{\partial C}{\partial A}dA - \frac{1}{2}\theta d\tau$$

Let F be a portfolio composed of one option, b units of H, and p units of A, then:

$$F = C + bH + pA$$

The dynamics of this portfolio are :

$$dF = dC + bdH + pdA$$

Choose b, p such that $b = -\frac{\partial C}{\partial H}$, $p = -\frac{\partial C}{\partial A}$, then:

$$dF = \left(\frac{\partial C}{\partial \tau} - \frac{1}{2}\theta \right) d\tau$$

If the portfolio F uses no wealth, then in equilibrium it should yield a zero return.

$$F = C - \frac{\partial C}{\partial H}H - \frac{\partial C}{\partial A}A \tag{3.68}$$

That is, if $F = 0$, then $dF = 0$ which implies that

$$\frac{\partial C}{\partial \tau} = \frac{1}{2}\theta \tag{3.69}$$

Look for a function $C(H, A, X, T)$ that solves equation (3.69) and is also subjected to the boundary conditions (3.61- 3.62). According to Grabbe (1983), the solution to the European call is given by:

$$C(t) = S(t)A^*(t, T)N(d_1) - XA(t, T)N(d_2) \tag{3.70}$$

where N(d_i) is the cumulative standard normal distribution with mean 0 and variance 1 and

$$d_1 = \frac{\ln(\frac{SA^*}{XA}) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(\frac{SA^*}{XA}) - \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}$$

$$\sigma^2 = \int_0^T \frac{1}{T}[\sigma_H^2(t + T - u, u) + \sigma_A^2(t + T - u, u)$$

$$- 2\rho_{HA}(t + T - u, u)\sigma_H(t + T - u, u)\sigma_A(t + T - u, u)]du$$

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But equation (3.69) shall be solved numerically and compared with that of stochastic correlation.

3.4.1 Stochastic correlation

Consider equation (3.66) and (3.65)

$$\frac{dH}{H} = \mu_H(t,T)dt + \sigma_H(t,T)dW_4$$

$$\frac{dA}{A} = \mu_A(t,T)dt + \sigma_A(t,T)dW_3$$

with a correlation coefficient between them

$$dW_3 dW_4 = \rho dt$$

where

$$d\rho = a(m-\rho)dt + c\sqrt{1-\rho^2}dW_5 \qquad (3.71)$$

$a(m-\rho)$ is the drift term, $c\sqrt{1-\rho^2}$ is the volatility term and the bound for correlation is $-1 \leq \rho \leq 1$. Assume

$$dW_4 dW_5 = \rho_1 dt$$

$$dW_3 dW_5 = \rho_2 dt$$

where ρ_1 and ρ_2 are constants.

The correlation matrix become $\begin{pmatrix} 1 & \rho & \rho_1 \\ \rho & 1 & \rho_2 \\ \rho_1 & \rho_2 & 1 \end{pmatrix}$ which must be positive definite that is

its determinant is zero or positive. Using Ito's Lemma, a three-dimensional stochastic

differential of the differential equations 3.66, 3.65 and 3.71 is obtained.

$$dC = \frac{\partial C}{\partial H}dH + \frac{\partial C}{\partial A}dA + \frac{\partial C}{\partial \rho}d\rho + (\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial H^2}H^2\sigma_H^2 + \frac{1}{2}\frac{\partial^2 C}{\partial A^2}A^2\sigma_A^2 + \frac{1}{2}\frac{\partial^2 C}{\partial \rho^2}\rho^2\sigma_\rho^2$$

$$+ \frac{\partial^2 C}{\partial H \partial A}HA\rho\sigma_H\sigma_A + \frac{\partial^2 C}{\partial H \partial \rho}H\rho_1\sigma_H\sigma_\rho + \frac{\partial^2 C}{\partial A \partial \rho}A\rho_2\sigma_A\sigma_\rho)dt \qquad (3.72)$$

where $\sigma_\rho = c\sqrt{1 - \rho^2}$

Assume that, under the risk-neutral measure Q, H and A are geometric Brownian motions with mean r (the risk-free interest rate) and constant volatilities $\sigma_H > 0$, $\sigma_A > 0$, with respect to Brownian motions W_4, W_3 satisfying;

$$\frac{dH}{H} = rdt + \sigma_H(t, T)dW_4 \qquad (3.73)$$

$$\frac{dA}{A} = rdt + \sigma_A(t, T)dW_3 \qquad (3.74)$$

Substituting equations (3.73), (3.74) and (3.71) in equation (3.72) gives;

$$dC = [\frac{\partial C}{\partial t} + \frac{\partial C}{\partial H}rH + \frac{\partial C}{\partial A}rA + \frac{\partial C}{\partial \rho}a(m - \rho) + \frac{1}{2}\frac{\partial^2 C}{\partial H^2}H^2\sigma_H^2 + \frac{1}{2}\frac{\partial^2 C}{\partial A^2}A^2\sigma_A^2 + \frac{1}{2}\frac{\partial^2 C}{\partial \rho^2}\rho^2\sigma_\rho^2$$

$$+ \frac{\partial^2 C}{\partial H \partial A}HA\rho\sigma_H\sigma_A + \frac{\partial^2 C}{\partial H \partial \rho}H\rho_1\sigma_H\sigma_\rho + \frac{\partial^2 C}{\partial A \partial \rho}A\rho_2\sigma_A\sigma_\rho]dt + \frac{\partial C}{\partial H}\sigma_H H dW_4$$

$$+ \frac{\partial C}{\partial A}\sigma_A A dW_3 + \frac{\partial C}{\partial \rho}\sigma_\rho dW_5 \qquad (3.75)$$

To obtain the price of the option, following the Black-Scholes analysis, consider two different options, $C_1(H, A, K_1, T_1)$ and $C_2(H, A, K_2, T_2)$ on H, A, ρ ;

Define a portfolio F by

$$F = C_1 + \lambda_1 C_2 + \lambda_2 H + \lambda_3 A$$

where λ_1, λ_2 and λ_3 are units

Assume that F is self-financing. It follows that the dynamics of this portfolio are :

$$dF = dC_1 + \lambda_1 dC_2 + \lambda_2 dH + \lambda_3 dA$$

$$dF = [\frac{\partial C_1}{\partial t} + \frac{\partial C_1}{\partial H}rH + \frac{\partial C_1}{\partial A}rA + \frac{\partial C_1}{\partial \rho}a(m-\rho) + \frac{1}{2}\frac{\partial^2 C_1}{\partial H^2}H^2\sigma_H^2 + \frac{1}{2}\frac{\partial^2 C_1}{\partial A^2}A^2\sigma_A^2 + \frac{1}{2}\frac{\partial^2 C_1}{\partial \rho^2}\rho^2\sigma_\rho^2$$

$$+ \frac{\partial^2 C_1}{\partial H \partial A}HA\rho\sigma_H\sigma_A + \frac{\partial^2 C_1}{\partial H \partial \rho}H\rho_1\sigma_H\sigma_\rho + \frac{\partial^2 C_1}{\partial A \partial \rho}A\rho_2\sigma_A\sigma_\rho]dt + \lambda_1[\frac{\partial C_2}{\partial t} + \frac{\partial C_2}{\partial H}rH$$

$$+ \frac{\partial C_1}{\partial A}rA + \frac{\partial C_1}{\partial \rho}a(m-\rho) + \frac{1}{2}\frac{\partial^2 C_2}{\partial H^2}H^2\sigma_H^2 + \frac{1}{2}\frac{\partial^2 C_2}{\partial A^2}A^2\sigma_A^2 + \frac{1}{2}\frac{\partial^2 C_2}{\partial \rho^2}\rho^2\sigma_\rho^2 + \frac{\partial^2 C_2}{\partial H \partial A}HA\rho\sigma_H\sigma_A$$

$$+ \frac{\partial^2 C_2}{\partial H \partial \rho}H\rho_1\sigma_H\sigma_\rho + \frac{\partial^2 C_2}{\partial A \partial \rho}A\rho_2\sigma_A\sigma_\rho]dt + \lambda_2 rHdt + \lambda_3 rAdt + [\frac{\partial C_1}{\partial H}\sigma_H H + \frac{\partial C_2}{\partial H}\lambda_1\sigma_H H$$

$$+ \lambda_2\sigma_H H]dW_4 + [\frac{\partial C_1}{\partial A}\sigma_A A + \frac{\partial C_2}{\partial A}\lambda_1\sigma_A A + +\lambda_3\sigma_A A]dW_3 + [\frac{\partial C_1}{\partial \rho} + \lambda_1\frac{\partial C_2}{\partial \rho}][\sigma_\rho]dW_5$$

For the portfolio F to be risk neutral, the factors in front of dW_3, dW_4 and dW_5 need to be zero. This can be achieved by letting λ_1, λ_2 and λ_3 to be;

$$\lambda_1 = -\frac{\partial C_1/\partial \rho}{\partial C_2/\partial \rho}$$

$$\lambda_2 = \frac{\partial C_1/\partial \rho}{\partial C_2/\partial \rho}\frac{\partial C_2}{\partial H} - \frac{\partial C_1}{\partial H}$$

$$\lambda_3 = \frac{\partial C_1/\partial \rho}{\partial C_2/\partial \rho}\frac{\partial C_2}{\partial H} - \frac{\partial C_1}{\partial H}$$

The choices of λ_1, λ_2, λ_3 above makes the portfolio risk neutral, so by absence of arbitrage it must hold that $dF = rFdt$. This means that

$$\Phi_1 dt - \frac{\partial C_1/\partial\rho}{\partial C_2/\partial\rho}\Phi_2 dt + \left(\frac{\partial C_1/\partial\rho}{\partial C_2/\partial\rho}\frac{\partial C_2}{\partial H} - \frac{\partial C_1}{\partial H}\right)rHdt + \left(\frac{\partial C_1/\partial\rho}{\partial C_2/\partial\rho}\frac{\partial C_2}{\partial H} - \frac{\partial C_1}{\partial H}\right)rAdt$$

$$= r\left[C_1 - C_2\left(\frac{\partial C_1/\partial\rho}{\partial C_2/\partial\rho}\right) + \left(\frac{\partial C_1/\partial\rho}{\partial C_2/\partial\rho}\frac{\partial C_2}{\partial H} - \frac{\partial C_1}{\partial H}\right)H + \left(\frac{\partial C_1/\partial\rho}{\partial C_2/\partial\rho}\frac{\partial C_2}{\partial H} - \frac{\partial C_1}{\partial H}\right)A\right]dt$$

(3.76)

where Φ_1, Φ_2 refer to the dt terms of dC_1 respectively dC_2.

Simplifying equation (3.76), gives;

$$\frac{\Phi_1 - rC_1}{\partial C_1/\partial\rho}dt = \frac{\Phi_2 - rC_2}{\partial C_2/\partial\rho}dt$$

(3.77)

Clearly the left-hand side of equation (3.77) does not depend on C_2, and the right-hand side does not depend on C_1, so both sides of the equation do not depend on C_1 and C_2, so are equal to a function $y(H, A, \rho, t) = \rho y$, which can be considered a premium for correlation risk. This tells us that the price process of a derivative C is a solution of the PDE

$$\Phi - rC - y\rho\frac{\partial C}{\partial\rho} = 0$$

(3.78)

where Φ is the dt term of dC. Writing equation (3.78) out fully gives us;

$$\frac{\partial C}{\partial t} + \frac{\partial C}{\partial H}rH + \frac{\partial C}{\partial A}rA + \frac{1}{2}\frac{\partial^2 C}{\partial H^2}H^2\sigma_H^2 + \frac{1}{2}\frac{\partial^2 C}{\partial A^2}A^2\sigma_A^2 + \frac{1}{2}\frac{\partial^2 C}{\partial\rho^2}\rho^2\sigma_\rho^2$$

$$+\frac{\partial^2 C}{\partial H\partial A}HA\rho\sigma_H\sigma_A + \frac{\partial^2 C}{\partial H\partial\rho}H\rho_1\sigma_H\sigma_\rho + \frac{\partial^2 C}{\partial A\partial\rho}A\rho_2\sigma_A\sigma_\rho - rC + [a(m-\rho) - y\rho]\frac{\partial C}{\partial\rho} = 0$$

(3.79)

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Since bonds are with zero coupon bonds, equation (3.79) becomes;

$$\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial H^2}H^2\sigma_H^2 + \frac{1}{2}\frac{\partial^2 C}{\partial A^2}A^2\sigma_A^2 + \frac{1}{2}\frac{\partial^2 C}{\partial \rho^2}\rho^2\sigma_\rho^2$$

$$+\frac{\partial^2 C}{\partial H\partial A}HA\rho\sigma_H\sigma_A + \frac{\partial^2 C}{\partial H\partial \rho}H\rho_1\sigma_H\sigma_\rho + \frac{\partial^2 C}{\partial A\partial \rho}A\rho_2\sigma_A\sigma_\rho + [a(m-\rho)-y\rho]\frac{\partial C}{\partial \rho} = 0$$

$$(3.80)$$

Equation (3.80) is valid for any option on foreign exchange with underlying measured in foreign currency but paid in domestic one. Since A is the only one which can be hedged, a solution independent of the exchange rate could be figured out. Rewriting the solution $C(H,A,t) = V(H,t)$ and letting $\tau = T - t$, gives:

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\frac{\partial^2 V}{\partial H^2}H^2\sigma_H^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \rho^2}\rho^2\sigma_\rho^2 + \frac{\partial^2 V}{\partial H\partial \rho}H\rho_1\sigma_H\sigma_\rho + [a(m-\rho)-y\rho]\frac{\partial V}{\partial \rho}$$

$$(3.81)$$

The payoff at expiration time $C(H,T) = \bar{A}\text{Max}[H(T) - X, 0]$, where $\bar{A}$ is a fixed exchange rate

Equation (3.81) is solved by finite difference methods that is the Crank-Nicolson method to increase the accuracy and stability of the solution .

3.4.2 Crank-Nicolson Method

Since equation (3.81) has three variables, three indices are employed. Let the time variable be indexed as i, H as j and ρ as k so that our equation is then discretized as;

$$\frac{V_{j,k}^{i+1} - V_{j,k}^i}{\Delta\tau} = \frac{H^2\sigma_H^2}{2}\left(\frac{1}{2}\left[\frac{V_{j+1,k}^{i+1} - 2V_{j,k}^{i+1} + V_{j-1,k}^{i+1}}{(\Delta H)^2} + \frac{V_{j+1,k}^i - 2V_{j,k}^i + V_{j-1,k}^i}{(\Delta H)^2}\right]\right)$$

$$+ \quad \frac{\rho^2\sigma_\rho^2}{2}\left(\frac{1}{2}\left[\frac{V_{j,k+1}^{i+1} - 2V_{j,k}^{i+1} + V_{j,k-1}^{i+1}}{(\Delta\rho)^2} + \frac{V_{j,k+1}^i - 2V_{j,k}^i + V_{j,k-1}^i}{(\Delta\rho)^2}\right]\right)$$

$$+ \quad H\rho_1\sigma_H\sigma_\rho\left(\frac{1}{2}\left[\frac{V_{j+1,k}^{i+1} - 2V_{j,k}^{i+1} + V_{j,k-1}^{i+1}}{\Delta H\Delta\rho} + \frac{V_{j+1,k}^i - 2V_{j,k}^i + V_{j,k-1}^i}{\Delta H\Delta\rho}\right]\right)$$

$$+ \quad [a(m-\rho) - y\rho]\frac{V_{j,k}^{i+1} - V_{j,k}^i}{\Delta\rho} \tag{3.82}$$

Equation (3.82) can be organized as;

$$\frac{V_{j,k}^{i+1}}{\Delta\tau} - \Big(\frac{H^2\sigma_H^2}{4(\Delta H)^2}(V_{j+1,k}^{i+1} - 2V_{j,k}^{i+1} + V_{j-1,k}^{i+1}) + \frac{\rho^2\sigma_\rho^2}{4(\Delta\rho)^2}(V_{j,k+1}^{i+1} - 2V_{j,k}^{i+1} + V_{j,k-1}^{i+1})$$

$$+ \frac{H\rho_1\sigma_H\sigma_\rho}{2\Delta H\Delta\rho}(V_{j+1,k}^{i+1} - 2V_{j,k}^{i+1} + V_{j,k-1}^{i+1}) + \frac{[a(m-\rho) - y]}{\Delta\rho}V_{j,k}^{i+1}\Big)$$

$$= \frac{V_{j,k}^i}{\Delta\tau} + \frac{H^2\sigma_H^2}{4(\Delta H)^2}(V_{j+1,k}^i - 2V_{j,k}^i + V_{j-1,k}^i) + \frac{\rho^2\sigma_\rho^2}{4(\Delta\rho)^2}(V_{j,k+1}^i - 2V_{j,k}^i + V_{j,k-1}^i)$$

$$+ \frac{H\rho_1\sigma_H\sigma_\rho}{2\Delta H\Delta\rho}(V_{j+1,k}^i - 2V_{j,k}^i + V_{j,k-1}^i) + \frac{[a(m-\rho) - \rho y]}{\Delta\rho}V_{j,k}^i$$

Now let

$$W = \Delta\tau\frac{H^2\sigma_H^2}{4(\Delta H)^2}, X = \Delta\tau\frac{\rho^2\sigma_\rho^2}{4(\Delta\rho)^2}, Y = \Delta\tau\frac{H\rho_1\sigma_H\sigma_\rho}{2\Delta H\Delta\rho}, Z = \Delta\tau\frac{[a(m-\rho) - \rho y]}{\Delta\rho}$$

such that;

$$V_{j,k}^{i+1} - [W(V_{j+1,k}^{i+1} - 2V_{j,k}^{i+1} + V_{j-1,k}^{i+1}) + X(V_{j,k+1}^{i+1} - 2V_{j,k}^{i+1} + V_{j,k-1}^{i+1})$$

$$+ Y(V_{j+1,k}^{i+1} - 2V_{j,k}^{i+1} + V_{j,k-1}^{i+1}) + ZV_{j,k}^{i+1}]$$

$$= V_{j,k}^{i} + W(V_{j+1,k}^{i} - 2V_{j,k}^{i} + V_{j-1,k}^{i}) + X(V_{j,k+1}^{i} - 2V_{j,k}^{i} + V_{j,k-1}^{i})$$

$$+ Y(V_{j+1,k}^{i} - 2V_{j,k}^{i} + V_{j,k-1}^{i}) + ZV_{j,k}^{i} \qquad (3.83)$$

3.4.3 Algorithms for solving the equations

In this subsection, the way of how to write and structure the programs required for solving the equations is considered. The program written for the numerical solution works as follows:

1. Define variables and ask user for input

2. If the user has chosen a Swaption, ask for spot rates and calculate value with which to multiply the final call solution

3. Resize matrices and vectors appropriately

4. Fill in τ, ρ and H vectors with step values

5. Fill in the initial values

6. Loop through all τ steps

   (i) Enter in boundary values

   (ii) Define entries of M (M is a tri-diagonal matrix)

   (iii) Define entries in right hand side vector

   (iv) Use the Gauss-Seidel iteration to calculate solution

   (v) Insert solution back into for V for that time step

(vi) Set previous solution to be new solution for use in the next time step

7. Output final row of solution to file to get prices for V at time $t = 0$.

3.4.4 Testing for stationarity

Exchange rates data and returns were tested for stationarity using the Augumented Dickey Fuller (ADF). The Dickey Fuller unit roots test are based on the following three regression forms:

1. With out constant and trend is $\triangle Y_t = \sigma Y_{t-1} + u_t$

2. With constant is $\triangle Y_t = \alpha + \sigma Y_{t-1} + u_t$

3. With constant and trend $\triangle Y_t = \alpha + \beta T + \sigma Y_{t-1} + u_t$

For Augmented Dickey Fuller (ADF) test, the following hypothesis is used:

$H_0 : \sigma = 0 (unitroot)$

$H_1 : \sigma \neq 0$

Decision rule

1. If the ADF statistic$>$ ADF critical value, it implies not to reject null hypothesis and hence implying non stationarity.

2. If the ADF statistic$<$ ADF critical value, it implies reject null hypothesis and hence implying stationarity.

3.4.5 Model Estimation

Under this study, the models will be estimated using Maximum Likelihood Estimation. The black scholes model under risk neutral probability is given by:

$$dS(t) = rS(t)dt + S(t)\sigma dW(t) \tag{3.84}$$

Using Itô lemma, the exact solution is given by:

$$S(t) = S(0) \exp[(r_t - \frac{1}{2}\sigma^2)t + \sigma W(t)] \qquad (3.85)$$

In the black scholes model, the unknown parameters are r and σ and denote them by $\theta = (r, \sigma)$. Before we estimate the parameters, we have to take the logarithm of the stock prices and calculate the returns of the Black Scholes Model which is given by:

$$R(t + dt) = \ln S(t + dt) - \ln S(t) \qquad (3.86)$$

$$(r - \frac{1}{2}\sigma^2)dt + \sigma W(t) \qquad (3.87)$$

where $s = 1$, $\mu = r - \frac{1}{2}\sigma^2$, $dt = 1$, $R(t + dt)$ is also a Geometric Brownian motion. The conditional mean and conditional variance are:

$$E[R(t + dt)|R(t)] = \mu \qquad (3.88)$$

$$Var[R(t + dt)|R(t)] = \sigma^2 \qquad (3.89)$$

and $R(t + dt) \sim N(\mu, \sigma^2)$.

The log-likelihood function is defined by:

$$\ln L(\theta) = -\frac{n}{2} \ln 2\Pi s - n \ln \sigma - \frac{1}{2} \sum_{t=1}^{n} \frac{R(t + dt) - \mu dt}{\sigma^2 dt} \qquad (3.90)$$

3.4.6 Model Selection

Model selection was done using the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC).

Akaike Information Criterion

The Akaike information criterion (AIC) is a measure of the relative quality of statistical models for a given set of data. Given a collection of models for the data, AIC estimates the quality of each model, relative to each of the other models. Hence, AIC provides a means for model selection. Let L be the maximized value of the likelihood function for the model; let k be the number of estimated parameters in the model. Then the AIC value of the model is given by:

$$AIC = 2k - 2\ln L \tag{3.91}$$

Given a set of candidate models for the data, the preferred model is the one with the minimum AIC value. Hence AIC rewards goodness of fit (as assessed by the likelihood function), but it also includes a penalty that is an increasing function of the number of estimated parameters. The penalty discourages overfitting (increasing the number of parameters in the model almost always improves the goodness of the fit).

Bayesian Information Criterion (BIC)

The Bayesian information criterion (BIC) is a criterion for model selection among a finite set of models. The model with the lowest BIC is preferred. It is based on the likelihood function and it is closely related to the Akaike information criterion (AIC) (AIC and BIC feature the same goodness-of-fit). When fitting models, it is possible to increase the likelihood by adding parameters, but doing so may result in overfitting. The BIC value of the model is given in equation 3.92,where n is the number of observations or the sample size.

$$BIC = k\ln n - 2\ln L \tag{3.92}$$

Chapter 4

Numerical Results

Data from the daily closing exchange rates of Kenya and South Africa was used which was got from OANDA (https://www.oanda.com/solutions-for-business/historical-rates/main.html) starting from 1st - January - 2010 to 31st - December - 2015 and in total 1837 observations. Matlab and R softwares were used. Exchange rates for Kenya was considered to be the domestic currency and South Africa, the foreign currency. In financial time series there are trends and the trends are nearly impossible to predict and difficult to characterize mathematically. So log-returns are usually analysed, that is the logged-value of todays value divided by the one of yesterday. Let p_t and p_{t-1} denote the closing exchange rate at the current time (t) and previous day $(t - 1)$ respectively, log returns or continuously compounded returns at any time are given by:

$$r_t = \log\left(\frac{p_t}{p_{t-1}}\right) \tag{4.1}$$

Table 4.1: **Descriptive statistics of returns**

Statistic	USD.KES	USD.ZAR
Maximum	0.04538	0.0736700
Minimum	-0.05215	-0.1136000
Mean	0.00004111	-0.0000063
Standard deviation	0.004345903	0.01012737
Kurtosis	44.01931	13.91015
Skewness	-0.3129154	-0.1390287

Table 4.1 presents the summary statistics for the daily closing exchange rates returns of kenya and South Africa . These include the mean, standard deviation, Kurtosis and skewness. Kurtosis is significantly greater than three which implies that they are heavily tailed which is characteristic of financial market data (All series display significant leptokurtic behavior as evidenced by the large kurtosis with respect to the Gaussian distribution). All returns series have an observation of 1836. They are all left skewed that is the left tail is longer and the mass of the distribution is concentrated on the right of the figure.

4.0.7 Stationarity Testing Results

The results of test is given in the table 4.2

Table 4.2: **ADF test results**

Series	t-statistic or critical value	USD.KES	USD.ZAR
Exchange rates	t- statistic	-0.757708	-1.707412
Exchange rates	1%	-3.431295	-3.431290
Exchange rates	5%	-2.861842	-2.891840
Exchange rates	10%	-2.566973	-2.566972
Returns	t-statistic	-13.17461	-63.60072
Returns	1%	-3.431295	-3.431290
Returns	5%	-2.861842	-2.891840
Returns	10%	-2.566973	-2.566972

From Table (4.2), the null hypothesis of a unit root for the two exchange rates series is accepted meaning that exchange rates series are non-stationary and the null hypothesis of a unit root for the two return series is rejected meaning that return series are stationary.
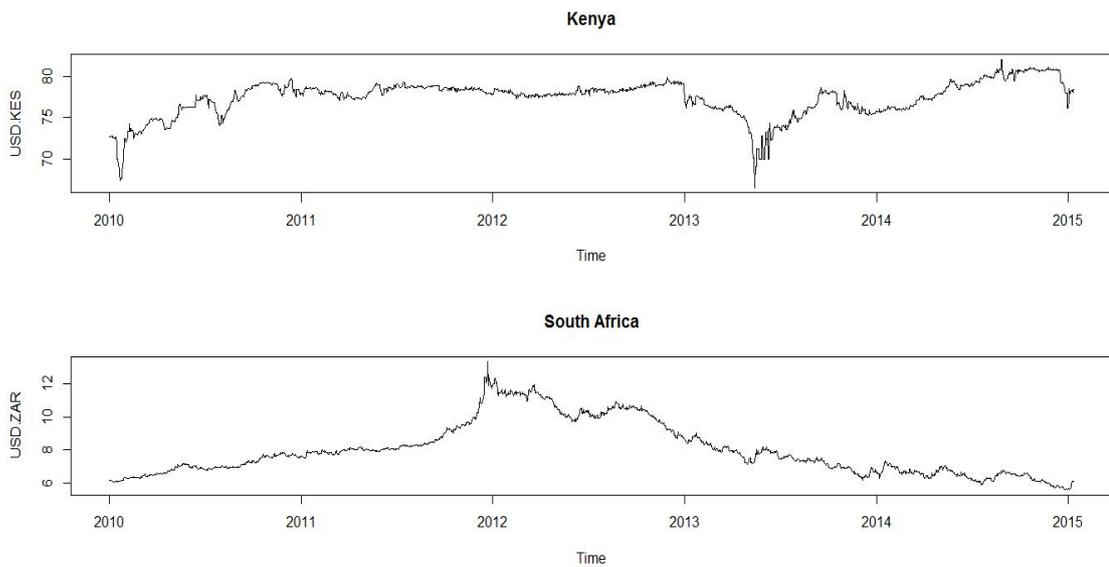

Figure 4.1: **Distributions of the exchange rates**

Figure 4.1 shows the evolution of the daily exchange rates that is for Kenya and South Africa respectively. Both series have trends (which implies that the mean is non constant). Generally, the trend of the Kenya exchange rates data exhibit a decline between 2010 and 2011 and between 2013 and 2014. However, the South Africa exchange rates data exhibit an upward trend in 2012. From a visual analysis, the graph reveals that there is a co-movement of the trends in a similar direction either upward or downward within the period under consideration.
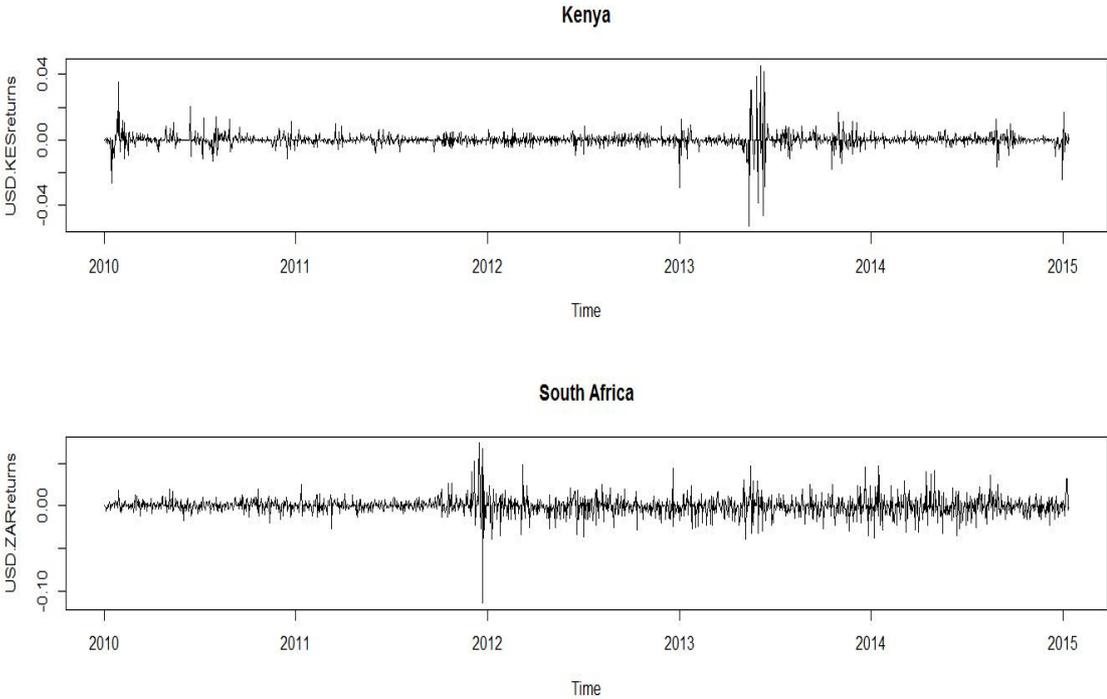

Figure 4.2: **Distributions of the returns**

Daily log returns on exchange rates data are presented in Figure 4.2 and exhibits no trends. The two graphs reveal the features of financial time series where volatility large clusters and asymmetric are evident.
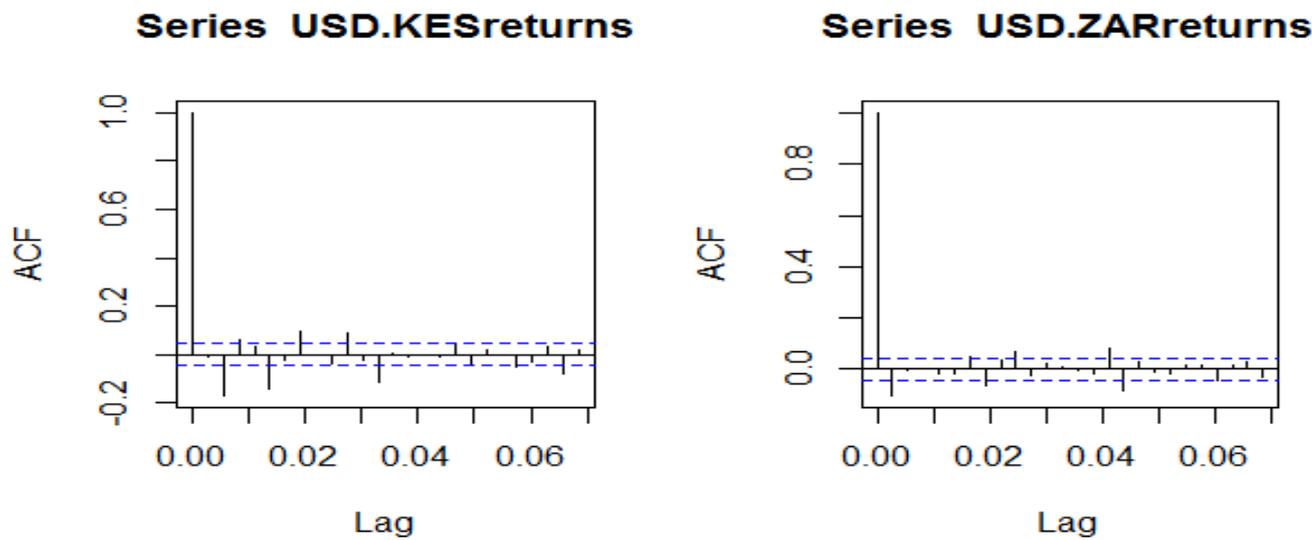
Figure 4.3: **Distributions for serial correlation**

Figure 4.3 presents the serial correlation of the returns. There is evidence of short serial correlation according to auto correlation function as seen clearly from Figure 4.3.

4.0.8 Testing for serial correlation

The BreuschGodfrey serial correlation LM test was done to test for serial correlation. The null hypothesis of the test is the there is no serial correlation in the residuals upto order 2.

Below is the output;

Breusch-Godfrey Serial Correlation LM Test:

F-statistic	525443.9	Prob. F(2,5840)	0.0000
Obs*R-squared	5811.703	Prob. Chi-Square(2)	0.0000

Test Equation:
Dependent Variable: RESID
Method: Least Squares
Date: 02/14/17   Time: 09:34
Sample: 1/01/2000 12/31/2015
Included observations: 5844
Presample missing value lagged residuals set to zero.

Variable	Coefficient	Std. Error	t-Statistic	Prob.
USD_KES	5.95E-05	1.74E-05	3.429947	0.0006
C	-0.004663	0.001371	-3.401828	0.0007
RESID(-1)	0.954572	0.013067	73.05132	0.0000
RESID(-2)	0.042808	0.013068	3.275918	0.0011

R-squared	0.994474	Mean dependent var	-2.97E-17
Adjusted R-squared	0.994471	S.D. dependent var	0.143197
S.E. of regression	0.010648	Akaike info criterion	-6.246188
Sum squared resid	0.662150	Schwarz criterion	-6.241620
Log likelihood	18255.36	Hannan-Quinn criter.	-6.244600
F-statistic	350295.9	Durbin-Watson stat	1.990521
Prob(F-statistic)	0.000000		

Figure 4.4: **Serial correlation testing for South Africa exchange rates**

The test rejects the null hypothesis of no serial correlation up to order two according to Figure 4.4. The Q-statistic and the LM test both indicate that the residuals are serially correlated this is because there probabilities is less than 0.05.

```
Breusch-Godfrey Serial Correlation LM Test:

F-statistic              250849.5   Prob. F(2,5840)          0.0000
Obs*R-squared            5776.756   Prob. Chi-Square(2)      0.0000


Test Equation:
Dependent Variable: RESID
Method: Least Squares
Date: 02/14/17   Time: 09:48
Sample: 1/01/2000 12/31/2015
Included observations: 5844
Presample missing value lagged residuals set to zero.

    Variable       Coefficient   Std. Error   t-Statistic    Prob.

    GBP_USD         -0.013428     0.055291     -0.242866    0.8081
       C             0.025616     0.092014      0.278392    0.7807
    RESID(-1)        0.859431     0.012964     66.29443     0.0000
    RESID(-2)        0.136145     0.012971     10.49574     0.0000

R-squared             0.988493   Mean dependent var       -2.00E-14
Adjusted R-squared    0.988488   S.D. dependent var        6.729499
S.E. of regression    0.722048   Akaike info criterion     2.187233
Sum squared resid     3044.702   Schwarz criterion         2.191801
Log likelihood       -6387.096   Hannan-Quinn criter.      2.188822
F-statistic           167233.0   Durbin-Watson stat        2.058057
Prob(F-statistic)     0.000000
```

Figure 4.5: **Serial correlation testing for Kenya exchange rates**

The test rejects the null hypothesis of no serial correhylation up to order two according to Figure 4.5. The Q-statistic and the LM test both indicate that the residuals are serially correlated this is because there probabilities is less than 0.05.

The following values of the parameters were used:

Table 4.3: **Parameters used**

σ_H	ρ_{HA}	σ_A	a	m	y
0.01	-0.0027	0.0043	$\frac{0.001}{1-\rho}$	1	2

Data was used to compute parameters such as σ_H, ρ_{HA} and σ_A. And parameters a, m and y were assumed after knowing the interval of the parameters of Cox-Ingersoll-Ross (CIR) process since our correlation dynamics is a CIR process.

The parameters above were used to solve equation (3.83) and the one for constant correlation (Black scholes model). The output is given in the Figures 4.6, 4.7 and 4.8:
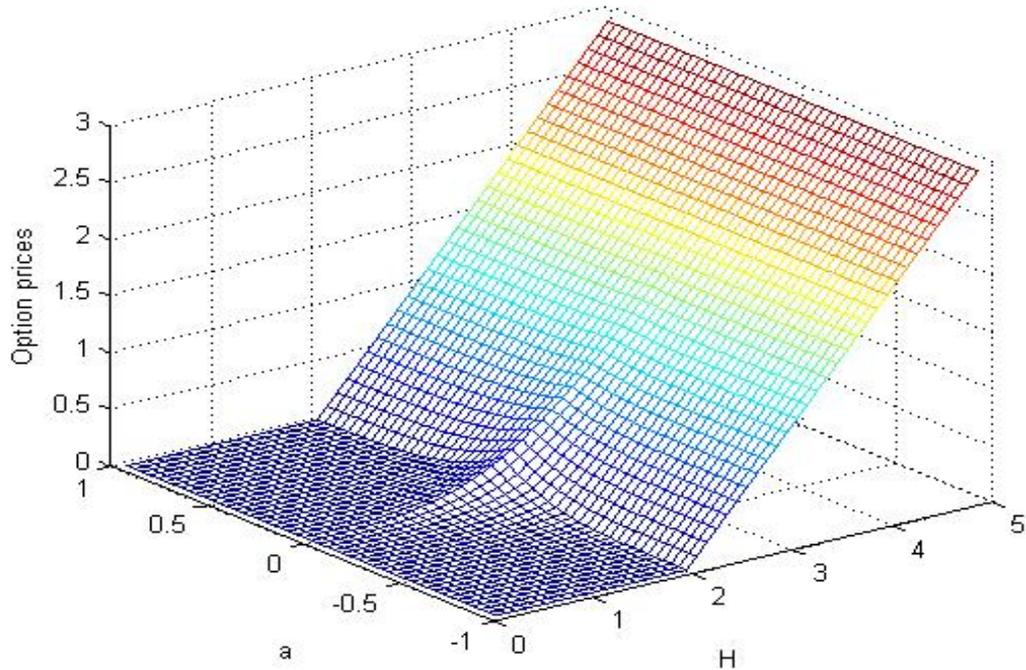

Figure 4.6: **Prices for European options under constant correlation**

Figure 4.6 gives the mesh for the prices of the European call for the Black scholes model at maturity time $T = 10$ which is not different from the known shape of the European call option prices where a $= A$ is the domestic currency price of a pure discount bond which pays one unit of foreign exchange at time $t + T$ and $H = S(t)A^*(t, T)$. $S(t)$ is the spot domestic currency price of a unit of foreign exchange at time t and $A^*(t, T)$ is the foreign currency price of a pure discount bond which pays one unit of foreign exchange at time $t + T$. It can be seen clearly that as H increases, the prices also increase which implies that prices depend on H.
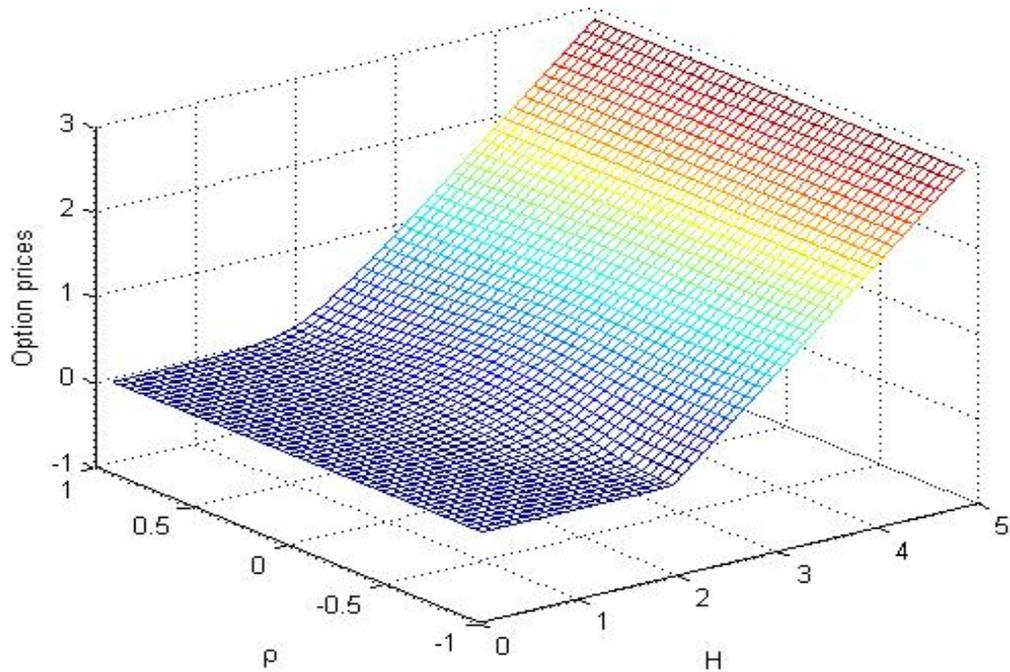
Figure 4.7: **Prices for European options under stochastic correlation**

Figure 4.7 gives the mesh for the prices of the European call for the modified model at maturity time $T = 10$ which is not different from the known shape of the European call option prices where ρ is the stochastic correlation and $H = S(t)A^*(t,T)$.

$S(t)$ is the spot domestic currency price of a unit of foreign exchange at time t and $A^*(t,T)$ is the foreign currency price of a pure discount bond which pays one unit of foreign exchange at time $t + T$. It can be seen clearly that as H increases, the prices also increase which implies that prices depend on H.
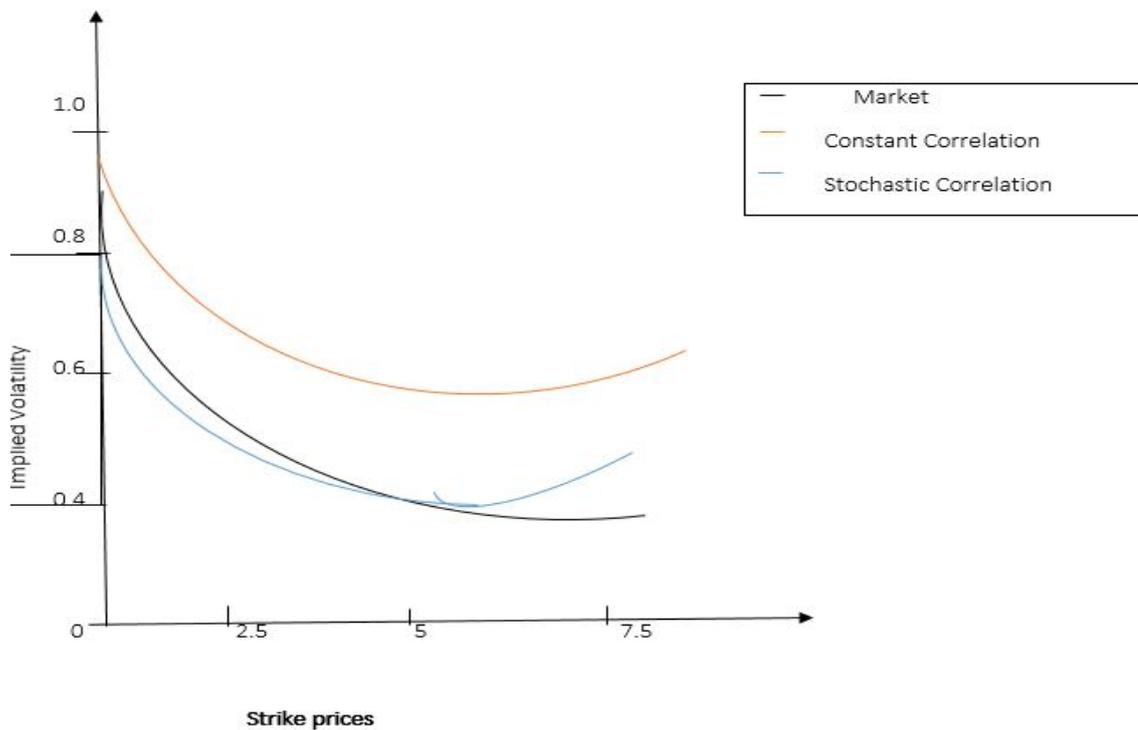
Figure 4.8: **The comparison of implied volatilities for the two models to the market volatilities of the Call-options.**

Implied volatilities for both models are compared to the market volatilities in Figure 4.8. It is observed that the implied volatilities for the modified model are much more closer to the market volatilities than the implied volatilities for the Black scholes model. The modified moodel provide better volatility smiles compared to the black scholes model.

4.0.9 Model Selection Results

Model selection was done using the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC).

And the table 4.4 gives the results:

Table 4.4: **Model Selection**

Model	$\ln L$	AIC	BIC
Stochastic Correlation	8311.865	-16619.73	-16608.7
Constant correlation	6767.971	-13531.94	-13520.91

From the table 4.4, model with stochastic correlation outperformed the model with constant correlation as observed from the smallest AIC and BIC values.

Chapter 5

Conclusion and Recommendation

5.1 Conclusion

The main objective of the study was to model volatility in emerging financial markets using Wishart Stochastic Volatility Model. This was divided into three specific objectives that is to construct a suitable Wishart Stochastic Volatility Model for emerging financial market, applying the newly constructed Wishart Stochastic Volatility Model in pricing of European call options and to compare results of a newly constructed Wishart Stochastic Volatility Model to the Black-Scholes model using real data from emerging markets. Pricing derivative in one and two dimension was discussed. Modified model was constructed basing on the characteristics of the emerging financial markets and the original Wishart Stochastic Volatility Model. Since Wishart Stochastic Volatility Model is a multidimensional model, $n = 2$ was considered. Foreign exchange derivative pricing was done for both constant and stochastic correlation where the prices for European call options for constant and stochastic correlation were derived numerically that is using the finite difference method called the Crank Nicolson method. Implied volatilities for both models was compared to the market volatilities in Figure 4.8 using exchange rates real data of Kenya and South

Africa Real data which was got from OANDA (https://www.oanda.com/solutions-for-business/historical-rates/main.html) starting from 1st - January - 2010 to 31st - December - 2015 and in total 1837 observations. Matlab and R softwares were used. Exchange rates for Kenya was considered to be the domestic currency and South Africa to be the foreign currency. The data was first tested statistically and graphically before it was used. It was found that returns were heavily tailed, stationary and had evidence of short serial correlation. Pricing equation for the European call with stochastic correlation provide better volatility smiles compared to the Black scholes model and model with stochastic correlation outperformed the model with constant correlation as observed from the smallest AIC and BIC values as given in table 4.4.

5.2   Recommendation

From the study, models with stochastic correlation need to be considered before those with constant correlation.

Further work need to be done in this area such as pricing derivatives for $n \geq 3$ where n represents the number of assets, using different maturity time when comparing the modified model with the black scholes model.

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